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AN ANALYSIS OF THERMALLY INDUCED FLOW OSCILLATIONS IN THE NEAR-CRITICAL AND SUPER-CRITICAL THERMODYNAMIC REGION

BY

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Abstract

Three mechanisms which can induce thermo-hydraulic oscillations at near-critical and at super-critical pressures are distinguished and discussed.

Experiments show that low frequency flow oscillations are most prevalent in systems of practical interest. A quantitative formulation and analysis is therefore presented concerned with predicting the onset of these "chugging" oscillations as function of fluid properties, system geometry and operating conditions.

The problem is analyzed by perturbing the inlet flow, linearizing the set of governing equations and integrating them along the heated duct to obtain the characteristic equation. The latter is given by a third order exponential polynomial with two time delays.

Conditions leading to aperiodic as well as to periodic flow phenonema are investigated. The first pertains to the possibility of <u>flow excursion</u> the latter to the onset of <u>flow oscillation</u>.

Stability maps and stability criteria are presented which, previously, were not available in the literature. They can be used to determine:

a) The region of stable and unstable operation and

b) The effects which various parameters have on either promoting or preventing the appearance of flow oscillations.

The effects of various parameters are analyzed and improvements are suggested whereby the onset of flow oscillation can be eliminated.

The similarity between "chugging" combustion instabilities and thermally induced flow oscillations at near and super-critical pressures is pointed out.

A review of the present understanding of the near-critical thermodynamic region is also presented.

Acknowledgments

The author wishes to express his appreciation to the NASA, George C. Marshall Space Flight Center, Huntsville, Alabama, for the support of this work and to Mr. Max Nein of this Center for his encouragement and assistance in the course of the research.

The author is indebted also to Dr. J. Boure' for the many fruitful and stimulating discussions on boiling instabilities during Dr. Boure's stay at the Research and Development Center of the General Electric Company.

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1. Research Objectives

This research was conducted to determine the fundamental nature of oscillation, and of instabilities in the flow of cryogenic fluids with heat addition.

The investigation was motivated by the fact that severe oscillations have been experienced in rocket engines heat exchangers utilizing oxygen and hydrogen at both subcritical and supercritical pressures.

The particular objectives of this investigation were:

1. To distinguish a number of mechanisms which may be responsible for thermally induced flow oscillations at near critical and at supercritical pressures.

2. To present a quantitative formulation of the mechanisms which appear to be most significant from the point of system design and operation.

3. To predict the onset of these oscillations in terms of the geometry and of the operating condition of the system.

4. To analyze the consequences of the theoretical predictions and to suggest improvements whereby the onset of these oscillations can be eliminated.

2. Summary and Conclusions

1. Mechanisms Leading to Unstable Operation

Three mechanisms which can induce thermo-hydraulic oscillations at near critical and at supercritical pressures have been distinguished.

One is caused by the variation of the heat transfer coefficient at the transposed i.e., at the pseudo-critical point.

The second is caused by the effects of large compressibility in the critical thermodynamics region.

Finally, the third mechanism is caused by variations of flow characteristics brought about by variations of fluid density during the heating process. The propagations of these variations through the system introduce various time delays which, under certain conditions, can cause unstable flow.

This last mechanism, which induces low frequency oscillations, was investigated in detail because available experimental data show that this type of flow oscillations is most prevalent in systems of practical interest.

2. Formulation of the Problem

The problem was formulated in terms of an equation of state and of three field equations describing the conservation of mass, energy and momentum.

The subcritical pressure range of operation was differentiated from the supercritical one by using the appropriate equation of state.

The problem was analyzed by perturbing the inlet flow, linearizing the set of governing equations and integrating them along the heated duct to obtain the characteristic equation.

3. <u>The Characteristic Equation</u>

The characteristic equation is given by a third order exponential polynomial with two time constants, (see Eq. V-15). It is expressed in terms of fluid properties, of system geometry and of operating conditions by means of influence coefficients (see Eq. V-16 through Eq. V-22).

The influence coefficients express the effects of the <u>inlet flow</u> perturbation and of the <u>space lag perturbation</u> on the various pressure drops of the system. By introducing various definitions for the average, for the log mean and for the mean densities and velocities it is shown that each pressure drop is weighed with respect to a different <u>velocity</u>. This result, which follows, from the integration of the governing set of equations, i.e., from the distributed parameter analysis, could not have been obtained from an analysis, based on "lumped" parameters. Consequently the accuracy of an analysis based on this latter approach can be estimated by means of the results obtained in this investigation.

The characteristic equation was used to obtain stability maps and stability criteria which, previously, were not available in the literature. The stability maps and criteria can be used to determine

a. The region of stable and of unstable operation and

b. The effects which various parameters may have on either promoting or on preventing the appearance of flow oscillations.

Conditions leading to aperiodic as well as to periodic flow phenomena were investigated. The first pertain to the possibility of <u>flow excursion</u> whereas the second pertain to the onset of <u>flow oscillation</u>. For this latter case the flow stability in systems with low inlet subcooling was considered separately from that corresponding to systems with high inlet subcooling.

The stability problem at intermediate subcooling will be considered in a future report.

4. Excursive Flow Instability

It was shown that, at supercritical pressures, a flow system with heat addition can undergo flow excursions because the hydraulic characteristics of the system are given by a cubic relation between the pressure and the mass flow rate (see Eq. VI-20). The latter is a consequence of density variations in the system.

This excursive flow instability, at supercritical pressures, is the equivalent of the "Ledinegg" excursive instability in boiling systems at subcritical pressures. This equivalence is supported by experimental data (see Figure VI-1) which shows that in both pressure regions, the flow system has similar hydraulic characteristics.

A stability criterion which predicts the onset of the excursive instability was derived in terms of system geometry, of fluid properties and of operating conditions, i.e., of system pressure, flow rate, inlet temperature and power input (see Eq. VI-13). Various aspects of this type of instability are discussed together with provisions required to prevent its appearance (see Section VI-2).

5. Flow Oscillations at Low Inlet Subcooling

It was shown that for a system with low inlet subcooling the characteristic equation can be reduced to a second order polynomial with one time delay (see Eq. VII-7). For such a system the propensity to flow oscillation can be analyzed by means of the stability maps recently presented by Bhatt and Hsu (see Figure VII-1).

It was shown further that, when the inertia can be neglected in a system with low inlet subcooling then the characteristic equation reduces to a first order exponential polynomial with one time delay (see Eq. VII-16).

For such a system the flow will be <u>unconditionally stable</u> if the stability number N_S (defined by Eq. VII-39) is larger than unity. If N_S is smaller than unity, stable operation is still possible <u>if</u> the angular frequency of the inlet perturbation is larger than the critical one (given in Eq. VII-40) or if the transit time is shorter than the critical one (given by Eq. VII-41).

The region of stable and of unstable operation are shown in a stability map (see figure VII-2) which can be used to analyze the effects that various parameters have on the propensity to induce or to prevent flow oscillations (see Section VII.3).

Although the analytical predictions have not yet been tested quantitatively, the trends predicted by this map and by the stability criterion (see Eq. VII-22 or Eq. VII-29) are in qualitative agreement with experimental observations (see Section VII.3).

6. Flow Oscillation at High Inlet Subcooling

It was shown that when the effects of the two time delays can be neglected then the characteristic equation reduces to a third order polynomial (see Eq. VIII-2). A stability criterion was also derived (see Eq. VIII-15) which specifies the conditions for stable operation.

Various aspects of this type of oscillations were discussed together with provisions required to prevent their appearance (see Section VIII-2). It is shown that the flow is more stable at high subcoolings than at low. Furthermore, it is concluded that the destabilizing effect of subcooling must go through a maximum at intermediate range, (see Section VIII-2).

7. Significance of the Results

The results of this analysis indicate several improvements in the design and/or in the operating conditions which can be made to prevent the onset of flow excursions or of flow oscillations. These are discussed in more detail in relation to each type of instability (see Sections VI-2, VII-3, and VIII-2).

It was shown that the predominance of a particular parameter results in a particular wave form and in particular frequency (see Eq. VII-40 and Eq. VIII-17). This result indicates that the primary cause of the instability can be determined from the trace of flow oscillations.

Perhaps the result of greatest significance revealed in the present investigation is the similarity between the characteristic equation which predicts "chugging" combustion instabilities and the characteristic equation which predicts the thermally induced flow oscillations for fluids in the near critical and in the supercritical thermodynamic region. Since it is well known that "chugging" combustion instabilities can be stabilized by an appropriate servo-control mechanism, the results of this investigation indicate that low frequency flow oscillations, at near critical and at supercritical pressures may be also stabilized.

The preceding conclusions have not yet been tested against experimental data. If confirmed, then the results of this study will provide a method whereby stable operation can be insured in an intrinsically unstable region.

3. Recommendations

The recommendations listed in the four tasks below, define the effort needed to complete and to verify the results obtained in this investigation.

1. Verify the stability criteria based on the second and first order exponential polynomials which have been derived in the course of these investigations. For this purpose use available experimental data for various fluids at subcritical and at supercritical pressures.

2. From the characteristic equation given by the third order exponential polynomial with two time delays (Eq. V-15) derive stability maps and stability criteria applicable to the entire range of subcoolings. Test these results against available experimental data.

3. Modify the characteristic equation to take into account the effects of the entire flow system i.e., of the flow loop. In particular include the effects of the inertia of the liquid in the storage tank and in the supply lines together with the flow and elastic characteristics of these lines.

4. Based on the results obtained from the preceding three tasks specify a servo-control mechanism which could be used to stabilize the flow for a system of practical interest and verify the results by experiments.

4. Nomenclature

MLTΘ System of Units

with H defined by $H = ML^2 T^{-2}$

A_c = cross sectional area of the duct $[L^2]$

A = coefficient defined by Eq. VII-16

a = coefficient defined by Eq. VII-10

a* = coefficient defined by Eq. VI-21

B = coefficient defined by Eq. VII-17

B* = coefficient defined by Eq. VI-4

b = coefficient defined by Eq. VII-11

b* = coefficient defined by Eq. VI-23

c = coefficient defined by Eq. VII-12

c* = coefficient defined by Eq. VI-23

cp = specific heat of the fluid in the "light" fluid region $[HM^{-1}\theta^{-1}]$

D = diameter of the duct $[L]$

f = friction factor

F_1 = Influence coefficient defined by Eq. V-5

F_2 = " Eq. V-6

F_3 = " Eq. V-7

F_4 = " Eq. V-8

F_5 = " Eq. V-9

F_6 = " Eq. V-10

F_7 = " Eq. V-11

G = Mass flux density $[ML^{-2}T^{-1}]$

I	=	Integrals given by Eq. IV-94, IV-97, IV-101, IV-107, IV-111.
i	=	Enthalpy $[HM^{-1}]$
Δi_{fg}	=	Latent heat of vaporization $[HM^{-1}]$
Δi_{21}	=	Inlet subcooling $[HM^{-1}]$
k_i	=	coefficient of the inlet flow restriction -
k_e	=	coefficient of the exit flow restriction -
ℓ	=	Total length of the heated duct $[L]$
M_f	=	Mass in the "heavy" fluid region per unit area $[ML^{-2}]$
M_g	=	Mass in the "light" fluid region per unit area $[ML^{-2}]$
N_s	=	Stability number defined by Eq. VII-39
P	=	system pressure $[ML^{-1}T^{-2}]$
ΔP_{ex}	=	Pressure rise of the external system $[ML^{-1}T^{-2}]$
ΔP_{o1}	=	Steady state pressure drop (SSPD) across inlet orifice defined by Eq. III-28 $[ML^{-1}T^{-2}]$
ΔP_{12}	=	SSPD due to friction in the heavy "fluid" region, defined by Eq. III-31 $[ML^{-1}T^{-2}]$
ΔP_{bf}	=	SSPD due to gravity in the heavy "fluid" region, defined by Eq. III-30 $[ML^{-1}T^{-2}]$
ΔP_a	=	SSPD due to acceleration in the "light" fluid region, defined by Eq. IV-89 $[ML^{-1}T^{-2}]$
ΔP_{bg}	=	SSPD due to gravity of the "light" fluid region, defined by Eq. IV-102 $[ML^{-1}T^{-2}]$

ΔP_{23} = SSPD due to friction of the light fluid region, defined by Eq. IV-112, $[ML^{-1}T^{-2}]$

ΔP_{34} = SSPD across exit flow restriction defined by Eq. IV-122. (M)

q = heat flux density $[HL^{-2}T^{-1}]$

$\dot{Q}$ = total heat input rate $[HT^{-1}]$

R = gas constant $[L^2T^{-2}\theta^{-1}]$

s = Exponent of the inlet velocity perturbation $[T^{-1}]$

S = Stability criterion defined by Eq VI-28

T = Period of the inlet velocity perturbation

t = time

u = velocity $[LT^{-1}]$

$\bar{u}_1$ = steady state velocity in the "heavy" fluid region LT^{-1}

$u_g(z)$ = S.S. velocity of the light fluid region defined by Eq. IV-28. (L)

$\bar{u}_3$ = S.S. velocity at the exit from the duct defined by Eq. IV-31. (L)

$\langle u_g \rangle$ = average velocity in the "light" fluid region defined by Eq. IV-32. (L)

$u_{\ell m}$ = Log mean velocity of the "light" fluid region defined by Eq. IV-36 (L)

u_m = mean velocity of the "light" fluid region defined by Eq. IV-38. (L)

δu_1 = inlet velocity perturbation given by Eq. III-7. (L)

δU_g = velocity perturbation of the "light" fluid region given by Eq. IV-30.

v_f = specific volume of the heavy fluid $[L^3M^{-1}]$

v_g = specific volume of the "light" fluid $[L^3M^{-1}]$

Δv_{fg} = change of specific volume in vaporization $[L^3M^{-1}]$

W = total steady state mass flow rate (MT^{-1})

z = length $[L]$

Green Letters

ξ = heated perimeter $[L]$

λ = space lag defined by Eq. III-20 $[L]$

$\delta\lambda$ = perturbation of the space lag $[L]$ defined by Eq. III-23.

ε = amplitude of inlet velocity perturbation $[LT^{-1}]$

τ_b = time lag, defined by Eq. III-18 $[T]$

$\tau_3-\tau_1$ = $\Delta\tau$ = total transit time, defined by Eq. III-63 $[T]$

τ_2 = critical transit time, defined by Eq. VII-41.

Ω = characteristic reaction frequency, defined by Eq. IV-21.

ρ_f = density of the "eavy" fluid $[ML^{-3}]$

ρ_g = density of the light fluid $[ML^{-3}]$

ρ_3 = density at the exit from the heated duct, defined by Eq. IV-72.

ρ_{lm} = log mean density in the light fluid region, defined by Eq. IV-76.

$\langle\rho_g\rangle$ = average density in the light fluid region defined by Eq. IV-73.

ρ_m = mean density in the "light" fluid region defined by Eq. IV-77.

ω = angular frequency of the inlet velocity perturbation T^{-1}

ω_c = critical angular frequency defined by Eq. VII-40.

ζ = dimensionless exponent defined by Eq. VII-8.

Subscripts

0, 1, 2, 3, 4 correspond to the locations of the duct indicated on Fig. II-2.

5. List of Illustrations

Page

I. Introduction

I.1 The Problem and Its Significance

A fluid in the vicinity of the critical point is an efficient heat transfer medium because of the large specific heat and of the large coefficient of thermal expansion. Consequently, the demand for increased efficiency of several advanced systems generated an interest in employing fluids at critical and supercritical pressures either as cooling or working media. For example, nuclear rockets, power reactors, high pressure once-through boilers, regenerative heat exchangers for rocket engines and a new sea water desalinization process are designed to operate in the critical and the supercritical thermodynamic region. These developments made it necessary to obtain data on and to improve the understanding of the thermal and the flow behavior over a broad range of fluid states.

A great number of investigations conducted for such a purpose have revealed that, in the critical as well as in the supercritical thermodynamic region, flow and pressure oscillations may occur when certain operating conditions are reached. These oscillations were observed in systems with forced flow as well as with natural circulation.

The occurrence of sustained pressure and flow oscillations and the attendant temperature oscillations are very undesirable and detrimental to reliable operation of a system. Mechanical vibrations and thermal fatigue induced by these oscillations very often result in a rupture of the duct. In liquid propellant rocket engines flow and pressure oscillations can also induce combustion instabilities resulting in a breakdown of the system.

Furthermore, in nuclear reactor systems flow and pressure oscillations may induce divergent power oscillations leading to the destruction of the entire system. Consequently, there is considerable practical interest and incentive to investigate, quantitatively, the conditions leading to the inception of these oscillations.

I.2 Previous Work

Severe pressure and flow oscillations were observed in experiments performed with various fluid in the supercritical thermodynamic region. Such oscillations were reported by Schmidt, Eckert and Grigull [1], (ammonia); Goldman [2, 3], (water); Firstenberg [4], (water); Harden [5], (Freon-114); Harden and Boggs [6], (Freon-114); Walker and Harden [7], (water, Freon-114, Freon-12, carbon dioxide); Holman and Boggs [8], (Freon-12); Hines and Wolf [9] (RP-1 and diethycyclohexane); Platt and Wood [10] (oxygen); Ellenbrook, Livingood and Straight [11], (hydrogen); Thurston [12], (hydrogen, nitrogen); Shitzman [13, 14] (water); Semenkover [15] (water); Cornelius and Parker [16] (Freon-114); Cornelius [17] (Freon-114); and Krasiakova and Glusker [18] (water).

For a given fluid the characteristics (frequency and amplitude) of these oscillations varied with operating conditions. In general, two types of oscillations were observed: acoustical and chugging oscillations. For example, Shitzman [13] reports that, for water at 250 atm, the pressure and temperature oscillations had a period of 80 sec. and an amplitude of 25 atm., and of 100°C respectively. Decreasing the flow rate and the power density resulted in decreasing the period to 15 sec. However, at a pressure of 5000 psi, Goldman [2, 3] reports pressure oscillations with frequencies from 1400 to 2200 cps. Similar high frequency oscillations (1000-10,000 cps and 380 psi peak to peak) were reported by Hines and Wolf [9] for RP-1.

Three classes of pressure oscillations in the supercritical region were observed in the experiments of Thurston [12]; these were described as:

1) Open-open pipe resonance observed at medium and high flow rate. This mode is associated with the fundamental wavelength of an open-open pipe.

2) Helmholtz resonance, associated with a resonator composed of a cavity connected to an external atmosphere via an orifice or neck.

3) Supercritical oscillations appearing usually at low flow rates.

Hines and Wolf [9], however, report only two general types of oscillations: a high frequency (3000-75000 cps) oscillations audible as a clear and steady scream and an oscillation with a lower frequency (600-2400 cps) which was audible as a chugging or pulsating noise. The dominant frequencies of these oscillations did <u>not</u> correspond to simple acoustic resonant frequencies for the tubes.

Cornelius and Parker [16, 17] describe in detail the two types of oscillations and note that the frequency of the acoustical oscillations decreases with temperature whereas the frequency of the chugging oscillations increases with temperature. Occasionally, both types of oscillations occured simultaneously.

A quantitative formulation and explanation of the conditions leading to the appearance of the pressure and flow oscillations has not been reported yet, although several qualitative explanations have been advanced. It is generally agreed that the oscillations are caused by the large variations of the thermodynamic and the transport properties of the fluid as it passes through the critical thermodynamic region.

Several investigators (12, 13, 14) note that the appearance of oscillations occurs when the temperature of the heating surface exceeds the "pseudo critical" or the "transposed" critical temperatures, i.e., the temperature where the specific heat reaches its maximum value (see Figure A1 in Appendix A). Oscillations were not observed if the inlet temperature was above this temperature. From this it was concluded that the mechanism for driving the oscillation occurred only when a "pseudo liquid" state was present in some parts of the heated duct.

Firstenberg (4) attributes the oscillations to the variations of the heat transfer rates to the fluid, whereas Goldman (2, 3) explains the oscillations as well as the steady state heat transfer mechanism in the critical and supercritical thermodynamic region as "boiling like" phenomena associated with non-equilibrium conditions. According to Goldman, below the pseudo-critical temperature the fluid is essentially a liquid, above this temperature it behaves as a gas. At the pseudocritical temperature, the density gradient and the specific heat reach maximum values giving an indication of the energy required to overcome the mutual attraction between the molecules. The fluid in the immediate vicinity of the heated wall is in a gas-like state; whereas the bulk fluid may still be in the liquid-like state. If by means of turbulent fluctuations a liquid-like cluster is brought into contact with the heating surface a large amount of energy will flow from the surface to the cluster because of the large temperature difference and because of the high conductivity of the liquid-like fluid. This energy may be large enough to "explode" clusters of molecules from the liquid-like state to the gas-like state. Thus, according to Goldman (2, 3), one may visualize the supercritical region as a region where explosions of liquid-like clusters into gas-like aggregates take place. Goldman considers this process to be similar to the formation of bubbles in liquids during boiling at subcritical pressures.

The conditions under which oscillations occur were summarized by Goldman as follows:

1) Heat transfer with "whistle" (i.e., with oscillations) occurs only at high heat flux densities and with bulk temperatures lower than the pseudocritical temperature.

2) At a given flow rate and inlet temperature, whistles occur at higher flux densities for higher pressure levels.

3) At given flow rate and pressure, whistles occur at lower heat flux densities for higher inlet temperatures.

4) At a given pressure and inlet temperature, whistles occur at higher heat flux densities for higher flow rates.

5) Whistles can be produced with various lengths of the test section, but the heat flux or inlet temperature must be increased to bring it about if the tube is shortened.

Visual observation that boiling-like phenomena can exist at supercritical pressures was reported by Griffith and Saberski [19] in experiments conducted with R-114 . The photographs of the process revealed a behavior similar to that observed in pool boiling at subcritical pressures.

Similarly, high speed movies of hydrogen at supercritical pressures taken by Graham, et al [20] revealed a phenomenon resembling boiling. Clusters of low density fluid were observed rising through a denser fluid-giving boiling-like appearance.

Hines and Wolf [9] attribute the appearance of the flow oscillations at supercritical pressures to the variations of liquid viscosity. They note that a small change of temperature near the critical point results in a large change of viscosity. Consequently, a sudden increase in wall temperature could cause a thinning of the laminar boundary layer due to variation of the viscosity. Thinning of the boundary layer would result in a drop of the wall temperature and a corresponding increase of viscosity. This would cause a thicker boundary layer and produce another rise of temperature, thus repeating the cycle. It was shown by Bussard and DeLauer [21] and by Harry [22] that a viscosity-dependent mechanism can induce an unstable flow in single phase flow systems when the absolute gas temperature is increased by a factor of 3.6 or more. Such flow oscillations were observed by Guevara et al [23] with helium flowing through a uniformly heated channel.

Harden and co-workers [5, 6, 7] concluded from their experiments that sustained pressure and flow oscillations appeared when the bulk fluid reached a temperature at which the product of the density and enthalpy has its maximum value. This explanation was, however, criticized by Cornelius [17].

Cornelius and Parker [16, 17] postulate that both acoustical and the chugging oscillations originate in the heated boundary layer. When the fluid in the heated boundary layer is in a "pseudo vapor" state, a pressure wave passing the heated surface would tend to compress the boundary layer, improve the thermal conductivity and cause an increase of the heat transfer coefficient. A rarefraction wave passing over the heated wall would have just the opposite effect. Thus, this pressure--dependent--heat-transfer rate could induce and maintain an acoustic oscillation. Cornelius and Parker attribute the appearance of chugging oscillations to "boiling-like" phenomena and a sudden improvement of the heat transfer coefficient. An approximate numerical solution verified the importance of the heat transfer improvement in triggering and maintaining oscillations.

Of particular interest to the analysis presented in this paper are the experimental results of Semenkover, [15] and of Krasiakova and Glusker [18], for water at 250 atm. For a constant power input $\dot{Q}$ to the system their data show a pressure versus mass flow relation that is illustrated in Figure I-1. It can be seen that for large values of inlet enthalpy i, there is a monotonic increase of pressure drop with flow rate. At a certain lower value of i, the curve shows an inflection point. For still lower values of inlet enthalpy, there is a region where the pressure drop decreases with increasing flow rate. Such a pressure drop-flow rate relation occurs in boiling systems and gives rise to an excursive type of instability which was analyzed first by Ledinegg [24] and by numerous investigators since [25 - 47]. Consequently, the data of [15, 18] tend to confirm the similarity between instabilities observed during subcritical boiling and those observed at supercritical pressure suggesting therefore a common mechanism.

I.3 Purpose and Outline of the Analysis

From the preceeding brief review of the present understanding of flow oscillations at supercritical pressures, it can be concluded that several modes of oscillation exist. It can be expected, therefore, that several mechanisms can be effective in inducing unstable flow. Indeed, as discussed in the preceeding section, several qualitative explanations of the phenomenon have been already advanced. However, a quantitative formulation of the problem is still lacking.

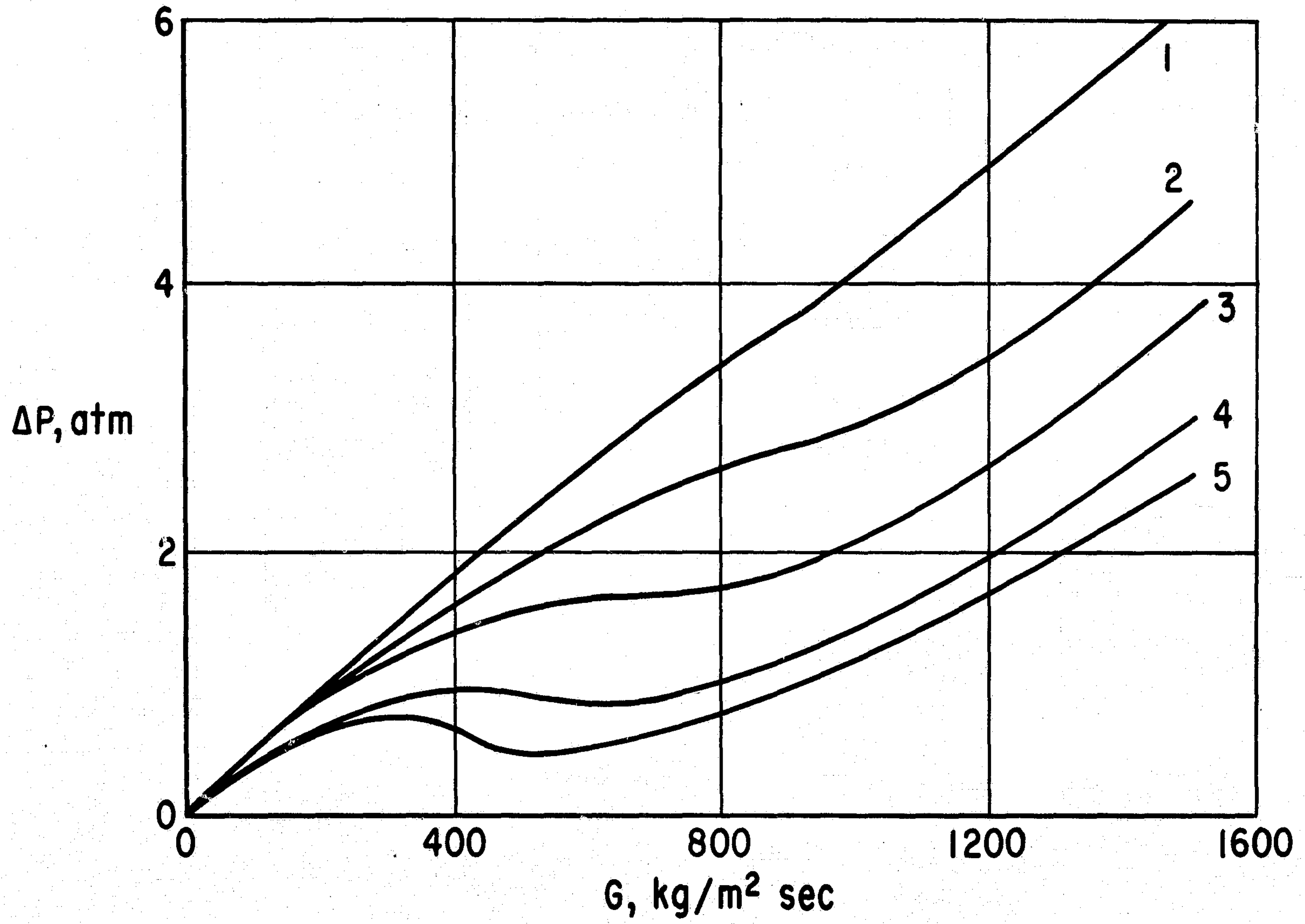

FIGURE I-1 HYDRAULIC CHARACTERISTICS FOR WATER AT SUPERCRITICAL PRESSURE (P = 250 atm) FLOWING THROUGH A HEATED DUCT. DATA OF SEMENKOVER FOR VARIOUS INLET ENTHALPIES: 1- 400, 2- 350, 3- 300, 4 - 200, 5- 100 IN kcal/kg.

The analysis presented in this paper has four objectives:

1) To distinguish a number of mechanisms which may be responsible for thermally induced flow oscillations at nearcritical and at supercritical pressures.
2) To present a quantitative formulation of the mechanism which appears to be most significant from the point of view of system design and operation.
3) To predict the onset of these oscillations in terms of the geometry and of the operating conditions of the system.
4) To analyze the consequences of the theoretical predictions and to suggest improvements whereby the onset of these oscillations can be eliminated.

The particular mechanism which is formulated and analyzed in this paper is based on the effects of time lag and of density variations. It is well known that these effects can induce combustion instabilities in liquid propellant rocket motors as discussed by Crocco and Cheng [48]. It was shown by Profos [49], Wallis and Heasley [50] and by Bouré [51] that the effects of time lag and of density variations can also induce unstable flow in boiling mixtures at subcritical pressures. The suggested similarity of flow oscillations observed at supercritical pressures with those observed in two phase mixtures at subcritical pressures prompts us to formulate and to analyze the problem in terms of this mechanism. In particular, the experimental results of Semenkover [15] and of Krasiakove and Glusker [18] discussed in the preceding section, together with the chugging oscillations described by several authors provide enough evidence to warrant a more detailed analysis of flow oscillations at supercritical pressures in terms of the time lag effect.

The present analysis is similar to those reported by Wallis and Heasley

[50] and by Bouré [51] in two respects: the formulation and the assumptions are the same. In particular, it is assumed that the density of the medium is a function of enthalpy only. The effects of pressure variations are, therefore neglected.* As noted by Wallis and Heasley [50] this assumption results in the decoupling of the momentum equation from the energy and the continuity equations. The momentum equation can be integrated then separately after the velocity and density variations are obtained from the continuity and the energy equations. Following Bouré [51] the problem is formulated in terms of an equation of state and of three field equations describing the conservation of mass, energy and mementum.

Apart from the fact that the analyses of Wallis and Heasley [50] and of Bouré [51] were derived to predict unstable flow in boiling two phase mixtures the present analyses (concerned with flow oscillations at nearcritical and at supercritical pressures) differs from theirs in two respects: 1) the form of the constitutive equation of state is different, 2) the characteristic equation describing the onset of oscillations is different. From this characteristic equation, we shall derive stability maps and stability inertia which, previously, were not available in the literature.

The outline of the paper is as follows. In Chapter II some general comments are made regarding 1) the nature of the thermally induced flow oscillations at nearcritical and at supercritical pressures, 2) the effect of the time lag, 3) the implication and limitations of the assumptions and 4) the general method of solution. In Chapters III and IV the problem is formulated and the set of governing equation is solved. The characteristic equation which predicts the onset of flow oscillation is derived in Chapter V; it is of the form of a third order exponential polynomial with two time delays. From the characteristic equation a criterion is derived in

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*The limitations and implications of this assumption are discussed in Chapter II.

Chapter VI which predicts the onset of an excursive type of instability at supercritical pressures.* This excursive instability at supercritical pressure is the equivalent of the so called Ledinegg excursive instability for boiling at subcritical pressures. The effect of time lags in inducing flow oscillations is analyzed in Chapters VII and VIII which consider first and second order expotential polynomials. Stability diagrams which predict the regions of stable and unstable flow in terms of the operating parameters are given in these two chapters together with suggested improvements whereby the onset of oscillations can be eliminated. The recommendations for future work and the conclusions are given in Chapters IX.

The status of the present understanding of thermodynamic phenomena that take place in the critical thermodynamic region is discussed in Appendix A.

I.4 The Significance of the Results

Three mechanisms which can induce thermo-hydraulic oscillations at supercritical pressures have been distinguished in this paper. One is caused by the variation of the heat transfer coefficient at the transposed, i.e., pseudo critical point. The second is caused by the effects of large compressibility and the resultant low velocity of sound in the critical region. Finally, the third mechanism is caused by the large variation of flow characteristics brought about by density variations of the fluid during the heating process. The propagations of these variations in particular of the enthalpy and of the density, through the system introduce delays which,

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*This criterion was first derived by the writer in the Second Quarterly Progress Report, "Investigation of the Nature of Cryogenic Fluid Flow Instabilities in Heat Exchangers," Contract NAS8-11422, 1 February 1965.

under certain conditions, can cause unstable flow. This last mechanism, that induces low frequency oscillations, is investigated in detail because experimental data show that this type of oscillation is most prevalent.

It is shown that at supercritical pressure unsteady flow conditions both excursive and oscillatory can occur. A characteristic equation is derived that predicts the onset of flow instabilities caused by density variations in the critical and supercritical thermodynamic region. The same characteristic equation can be used to predict the onset of flow instabilities in boiling at subcritical pressure, *if* the effect of the relative velocity between the two phases can be neglected. Experimental evidence shows that this effect becomes negligible at reduced pressures above say 0.85. Consequently, at near critical and supercritical pressures, the characteristic equation, which is expressed in terms of system geometry and operative conditions, can be used to determine:

a) The region of stable and unstable behavior.

b) The effect which various parameters may have on either promoting or on preventing the appearance of flow oscillations.

From this characteristic equation simple "rule of thumbs" criteria are also derived based on the assumption that one or the other of the various parameters is dominant. It is shown that the dominance of a particular parameter results in a particular frequency and wave form. This results permits a diagnosis of the primary cause of the instability from the trace of flow oscillations.

It is of particular interest to note that the characteristic equation derived in this paper for predicting flow oscillations at supercritical pressure is of the same form as a characteristic equation derived by Crocco and Cheng[48] to predict combustion instabilities of liquid propellant rocket motors.* It is well established in the combustion literature that a servo-control mechanism can be used to stabilize the low frequency combustion instability. The similarity of the characteristic equations is, therefore, significant because it indicates that stable operation could be insured also in the nearcritical and in the supercritical region by using an appropriately designed servo-control mechanism.

*This similarity between combustion and two phase flow instabilities should not come as a surprise if one recalls that the processes of combustion and of boiling are both chemical processes involving large enthalpy and density changes.

II. General Considerations

II.1 The System and the Thermodynamic Process

In order to understand the mechanisms of the thermally induced flow oscillations at supercritical pressures, it is necessary to examine briefly the system and the thermodynamic process.

The system of interest is shown in Figure II-1. It consists of a fluid flowing through a heated duct of length l. Without loss of generality it will be assumed that the duct is uniformly heated at a rate of $\dot{Q}$. Two flow restrictions are located one at the entrance, the other at the exit of the duct.

The thermodynamic process starts with the fluid at a supercritical pressure P, entering the heated duct with velocity u_0. The temperature T_0, of the fluid at the inlet is well below the critical temperature of the fluid under considerations. As the energy is being transferred from the heated duct to the fluid its temperature T, specific volume v, and enthalpy i, will increase. Thus, the temperature T_3, at the exit may be considerably above the critical temperature. In a number of systems of practical interest it can be assumed that this process takes place at an approximately constant pressure.

In order to formulate the problem it is necessary to specify the constitutive equation of state which describes the relation between say the specific volume, the pressure and the enthalpy for the particular fluid. This requires data on the thermodynamic properties of the fluid in the region of interest. The region of interest to this study are the nearcritical and the supercritical regions.

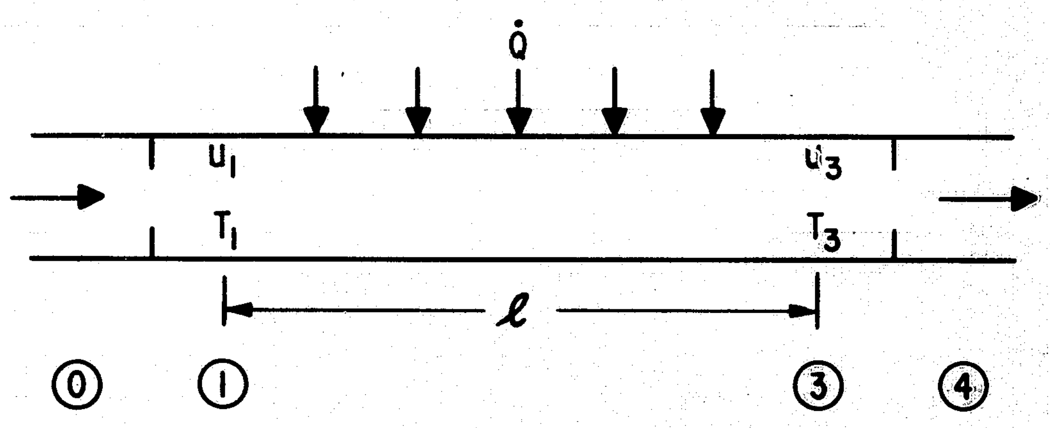

FIGURE II-1 THE SYSTEM

The present understanding of the thermodynamic properties and phenomena at nearcritical and supercritical pressures is reviewed in Appendix A. It is shown there that at these pressures the fluid has the characteristics of a liquid when the temperature is sufficiently below the critical one. However, if the temperature is increased sufficiently above the critical temperature, the fluid will have the characteristics of a gas. This is illustrated in Figure II-2 which is a plot of the specific volume and of the temperature versus the enthalpy for oxygen at a reduced pressure of $P_r = 1.1$.

It can be seen from this figure that at low enthalpies the specific volume is essentially constant, this is a characteristic of liquids. As the enthalpy increases the specific volume increases approaching values predicted by the perfect gas law. It can be seen also that this change from a liquid-like state (region 1 - 2) to a gas-like state (region 2' - 3) occurs over a transition region denoted by 2 - 2' on Figure II-2.

It appears, therefore, that at supercritical pressures the relation between the specific volume and the enthalpy can be approximated by considering three regions: a liquid-like, a transition and a gas like region. For oxygen Figure II-2 indicates also that, as a first approximation, the transition region can be reduced to a transition point reducing, therefore, the problem to a "two-region" approximation.* Since oxygen is the fluid of primary interest to this analysis, we shall use the "two-region"

*The "three region" approximation was first introduced by the writer in analyzing the excursive instability at supercritical pressures (see footnote on page 10). Following this work Dr. R. Fleming, from the Research and Development Center of the GE Co., introduced the "two region" approximation for oxygen. These two approximations are discussed further in Appendix B.

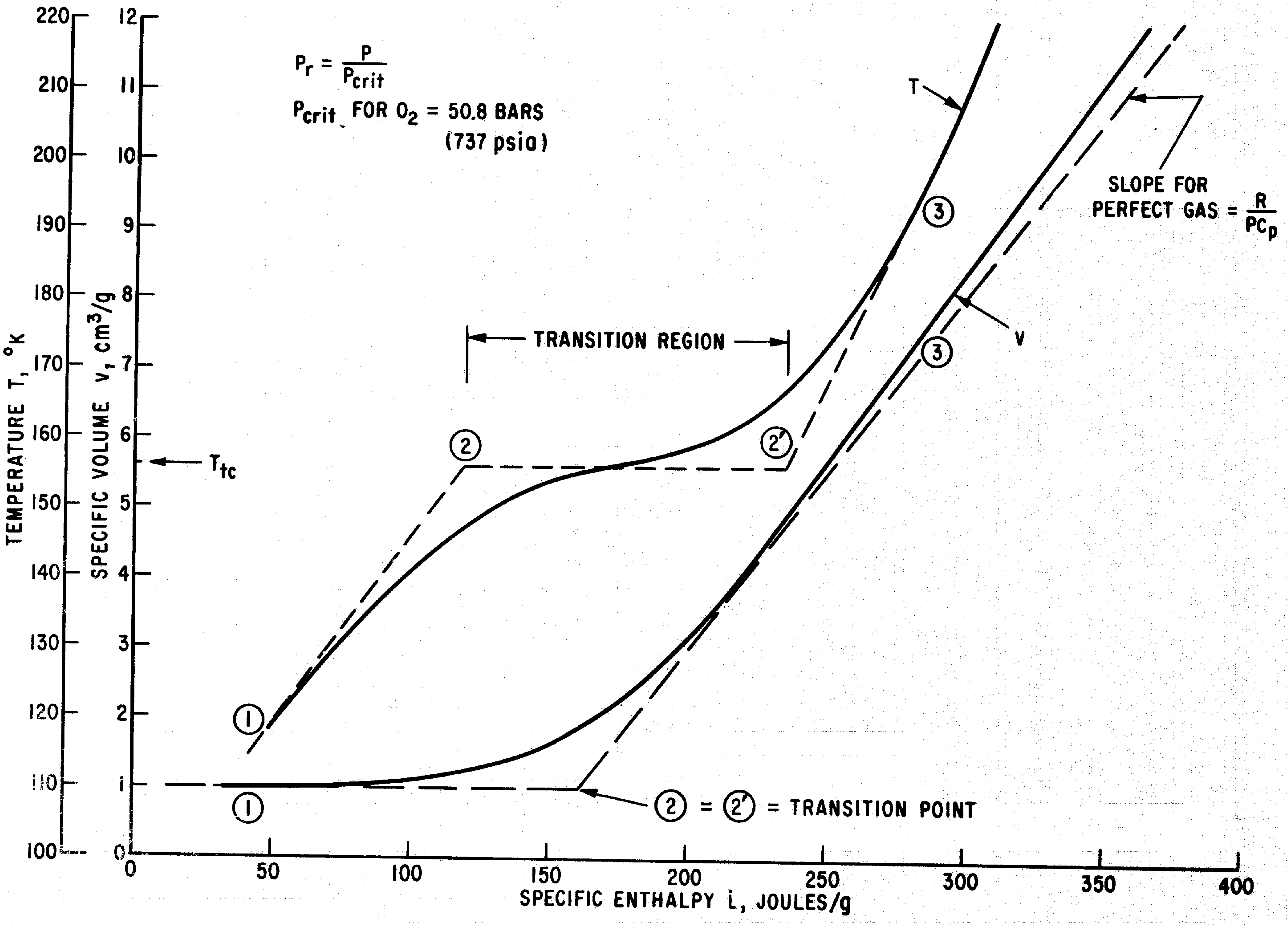

FIGURE II-2 SPECIFIC VOLUME AND TEMPERATURE VERSUS ENTHALPY FOR OXYGEN AT $P_r = 1.1$

approximation for describing the relation between the specific volume and the enthalpy. It is assumed, therefore, that the "heavy" fluid (of constant density) persists until the transition point is reached; above this point the fluid will have the properties of the 'light' phase. It remains now to define this transition point.

In boiling at subcritical pressure the transition from the heavy to the light phase corresponds to the onset of boiling. Consequently, it will be assumed that in the nearcritical region the transition point corresponds to the enthalpy at saturation temperature.

At supercritical pressures it will be assumed that the transition point corresponds to the transposed critical point, i.e., to the pseudo critical point which is defined as the point where C_p reaches its maximum value. It is discussed in Appendix A that the locus of pseudo critical points can be regarded as the extension of the saturation line in the supercritical region.

II.2 Time Lag and Space Lag

It is of interest to consider now the timewise and spacewise description of the process.*

If we follow a particle from the time it enters the heated section until it leaves it, we shall observe that its properties change from v_1

*We follow here Crocco and Cheng [48] who gave an equivalent description for combustion instabilities. The same comment applies to the three sections that follow. Indeed, this reference proved to be most stimulating and useful in the course of this investigation.

and i_1 at the inlet to v_3 and i_3 at the exit (see Figure II-3). In view of the "two-region" approximation we would note that the transition from the "heavy" to the "light" fluid occurred when the properties (specifically the enthalpy) reached values that correspond to the transition point. The time elapsed between the injection of the heavy particle in the heated duct and its transformation to the "light" fluid will be denoted as the <u>time lag</u> τ_b.

It is of interest also to consider the spacewise description of the process. In this case the time lag must be replaced by the <u>space lag</u> which is a vectorial quantity indicating the location in the duct where the transformation from the "heavy" to the "light" fluid takes place. The space lag is denoted by λ on Figure II-3. Of course, the space lag can be related to the time lag when the particle velocity is known.

Like in combustion, the location in the duct where the transformation takes place can be regarded as the source of the "light" fluid. It is obvious that the flow properties in the region occupied by the "light" fluid will depend upon the intensity of this source. If it is assumed that the injection rate of the "heavy" fluid is constant and that the time and space lags were constant, then the intensity of the source would also be independent of time resulting in a steady flow in the "light" fluid region. However, this is not the case because fluctuations which affect the time lag and/or the injection rate are present both at supercritical and at subcritical pressures. In the vicinity of the critical thermodynamic point large fluctuations of properties, in particular of density, are observed even in <u>non-flow</u> systems. In boiling systems fluctuations are always present because of variations of the rate of bubble formation

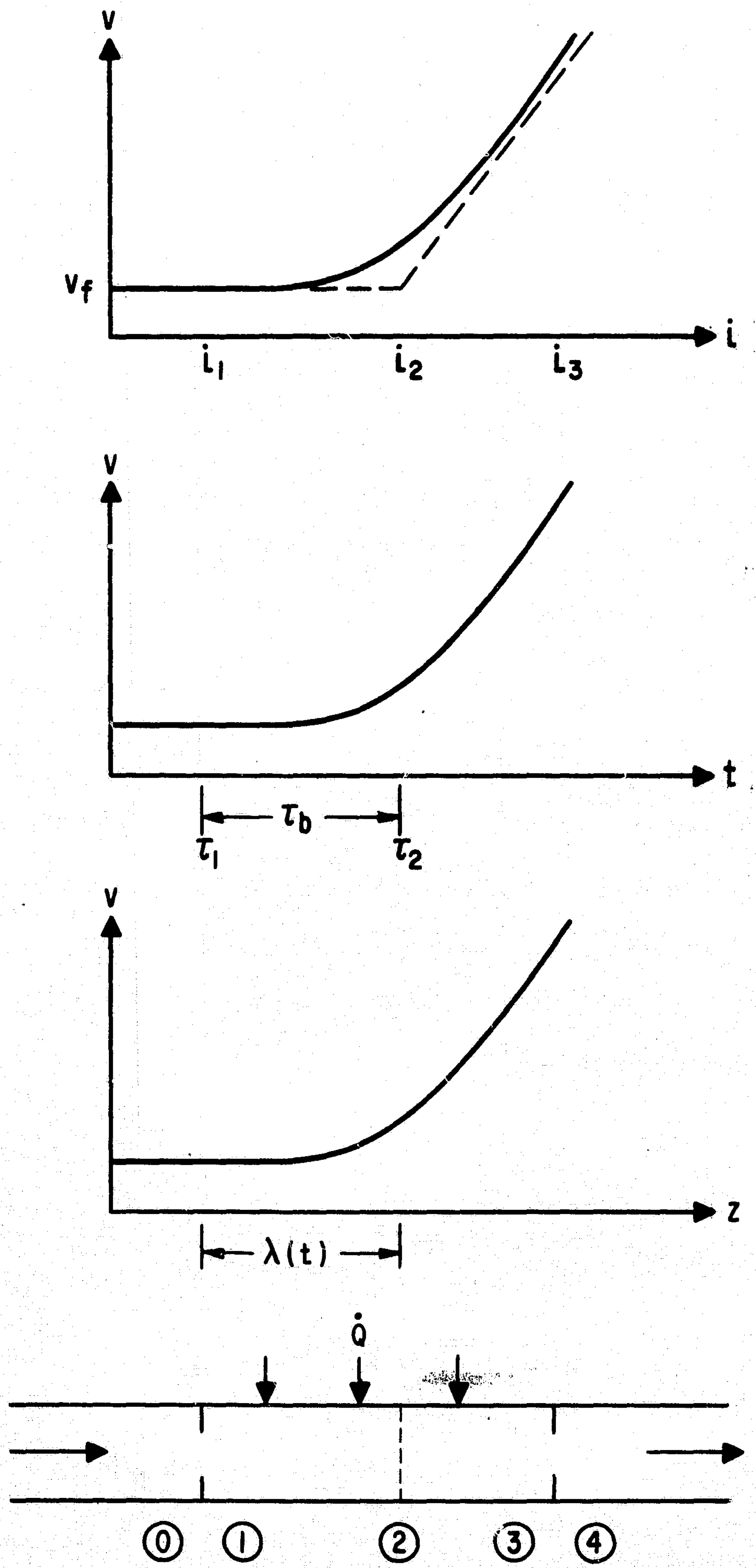

FIGURE II-3 "TWO-REGION" APPROXIMATION SHOWING THE TIME LAG AND THE SPACE LAG.

and population, of flow regimes, of the heat transfer coefficient, etc. Consequently, the strength of the source may fluctuate even when the injection rate is kept constant. It is evident also that variations of inlet velocity will introduce additional effects.

The nucleation and evaporation at subcritical pressures and the transformation of "heavy" clusters to "light" clusters at supercritical pressures are rate processes that occur during and have an effect upon the length of the time lag. Both of these transformation rates are affected by the pressure, temperature and by other rate processes such as the rate of energy transfer, flow rate, etc. If one of these factors changes or fluctuates, the transformation rates will fluctuate also resulting in a fluctuation of the time lag, i.e., in the fluctuation of the source. Since the source affects the flow conditions in the "light" fluid region the flow in this region may become oscillatory.

II.3 Organized Oscillations

Oscillations of a system can be always produced if properly excited. Such oscillations can be distinguished by a characteristic time, i.e., period if the process is periodic or by a relaxation time if it is aperiodic.

Like in combustion and following Crocco and Cheng [48] we shall distinguish two cases: random fluctuations and organized or coordinated oscillations.

As random fluctuation we consider those that are similar to fluctuations observed in ordinary turbulent flow. In this case it can be assumed that the transformation process, for example the rate of evaporation in boiling at subcritical pressures, is not excited. The fluctuations at one point do not have any effect on other fluctuations somewhere else in the system. Since the integrated effects of these fluctuations vanish they do not pose a problem.

In the case of an organized oscillation the transformation process will be excited by one or more coordinating processes such as the oscillation of the inlet flow rate, of the heat transfer coefficient, etc. The exciting force for maintaining the oscillation of the coordinating process is in turn provided by the transformation process. For example, in boiling systems oscillations of pressure will affect the saturation temperature which may induce oscillations in the rates of evaporation. These in turn may induce flow oscillations and provide the excitation force for maintaining the pressure fluctuations.

The fundamental character of organized oscillations is that a well defined correlation exists between fluctuations at two different points or instants. In other words that a disturbance is propagated, i.e., displaced in time and space through the system. When these organized oscillations are present their integrated effect does not vanish whence the interest in these oscillations. Furthermore, an oscillatory system may become unstable, i.e., it may have the tendency to amplify. In the example cited above pressure fluctuations of an increasing amplitude may be generated leading to the destruction of the system. When the effects are proportional to the causes the system is defined as linear. In this paper we are interested in such systems.

II.4 The Mechanism of Low Frequency (Chugging) Oscillation at Supercritical Pressures

It was noted in the preceding section that the characteristic of organized oscillations is the propagation of disturbances through one system. These disturbances can be variations of density, pressure, enthalpy, entropy, etc. In this section we shall examine the effects which these propagations may have on the oscillating propensity of the system. In particular, we shall consider the propagation of density disturbances and the effect of the time lag, i.e., of the space lag. The effects of pressure waves are discussed in Section II-7 together with the other mechanisms which may induce flow oscillations in the nearcritical and supercritical regions.

We note that the effect of the time lag in inducing combustion instabilities was already analyzed by Summerfeld [52], Crocco and Cheng [48] among others. In boiling systems, this effect was already analyzed by Profos [49], Wallis and Heasley [50] and by Bouré [51]. In these analyses the flow was assumed to be homogeneous, i.e., the effect of the relative velocity between the gas the liquid phase was neglected. A density propagation equation, applicable to two-phase mixtures, which takes this effect into account was formulated in [53] and solved in [54, 55].

Let us examine now the effects of the finite rates of propagation and of the resulting time lag and time delays on the flow in a system consisting of a constant pressure tank connected by a feeding system to the heated duct.

Consider first the tank and the feeding system only and let us perturb suddenly the inlet flow. If there is no feedback between the heated unit and the upstream part of the system, the steady state conditions will be restored. In particular, if the variation of the flow rate is small during

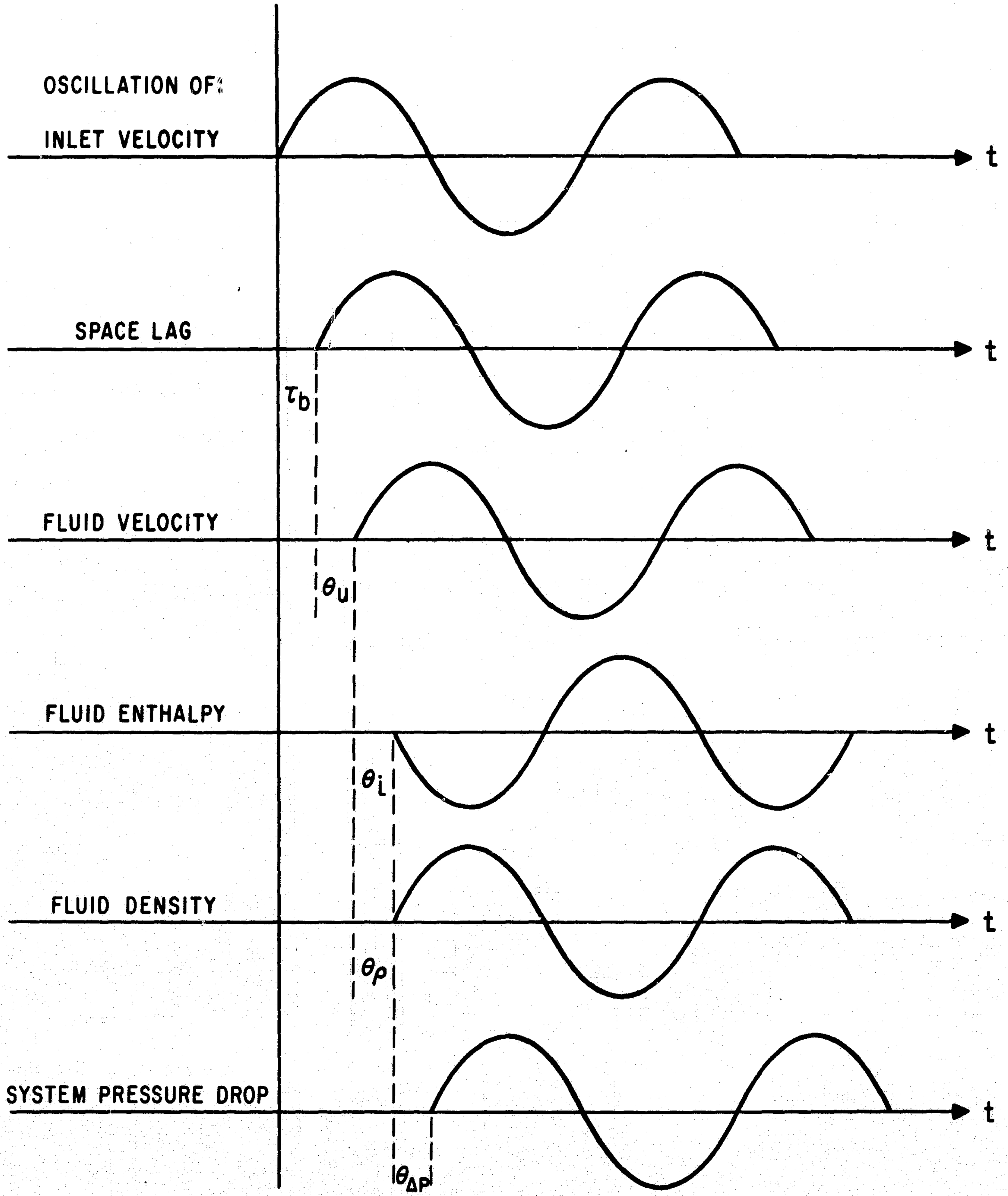

FIGURE II-4 THE EFFECTS OF TIME LAG AND OF SPACE LAG IN INDUCING FLOW OSCILLATION.

the time a pressure wave propagates back and forth through the tank and the feed system, then the pressure effects can be neglected. As discussed in [48] the process can be described then, with sufficient accuracy, by an exponentially decaying flow which is characterized by the relaxation time constant i.e., by the line relaxation time. Therefore, the system is stable because the steady state conditions will be eventually restored.*

Consider now the effect of a perturbed flow at the inlet of the heated duct (See Figure II-4). It is obvious that an oscillatory flow at the inlet will induce an oscillatory flow of the fluid in the duct. However, in absence of a driving mechanism these oscillations would also exhibit an exponential decay. We are looking for a mechanism whereby these flow oscillations at supercritical pressures can be maintained. Like in boiling and in combustion such a mechanism is provided by the propagation phenomena which introduce different delay times in the response of the system. This is shown in Figure II-4.

It can be seen on Figure II-4 that an oscillatory inlet flow can induce oscillations of the space lag; this is in accordance with the discussion of the preceding section. The onset of these oscillations is delayed however by the lag time τ_b, because of the finite rate of propagation of the disturbance. An oscillatory space lag, which is equivalent to an oscillatory source, will induce flow oscillation, in the "light" fluid region. These source-induced oscillations will be present in addition to those already induced by the inlet flow. Because of these two oscillatory motions there will be a delay time Θ_u in the flow response. Oscillations of the flow

*We are assuming here that any servo-control mechanism in the feeding system will not have a destabilizing effect.

will induce oscillations of enthalpy and of density both delayed by a certain delay time. With flow and density oscillating, the pressure drop in the duct will also oscillate. If the conditions are such that the minimum pressure drop in the duct occurs when the inlet flow is maximum, it is apparent then that the oscillations can be maintained. It is also obvious that whether or not this will occur will depend on the time lag τ_b and on the delay times θ_u, θ_ρ, $\theta_{\Delta P}$. When these delay times do <u>not</u> depend upon τ_b, it can be seen that increasing the lag time τ_b has a destabilizing effect. Since the time lag τ_b (see Figure II-4) depends upon the enthalpy difference $i_2 - i_1$, it can be concluded that, for this particular case, a decrease of inlet enthalpy i_1, i.e., that an increase of Δi_{21} has a destabilizing effect.

From this qualitative description it can be already seen that at supercritical pressures an unstable flow can be induced by the delayed response of various perturbations. It remains now to advance a qualitative description. We shall do this in the following chapters by modifying and applying the method proposed in [50, 51] for boiling at subcritical pressures.

II.5 <u>Method of Solution</u>

In what follows we shall consider the "heavy" and the "light" fluid regions separately. Each will be described in terms of three conservation equations and the equation of state. We shall use the one dimensional representation and obtain solutions for each region. These solutions will be matched at the transition point, i.e., at the end point of the space lag to provide a solution valid for the entire system.

Following [48, 50, 51] it will be assumed that the variation of pressures can be neglected. This is implied by the assumption that the density is function of enthalpy only. It can be seen that this assumption will be valid only if the variations of flow, density, enthalpy, etc. are relatively small during the total time for propagation of a pressure wave back and forth through the duct. Under this condition it can be assumed that the various disturbances move through a uniform medium. It is apparent also that this will be true only if the rate of propagation of pressure waves is considerably faster than the rate of propagation of the disturbances. However, both in boiling systems as well as in the nearcritical region the velocity of sound reaches very low values.* Consequently, it can be expected that there will be a range of operating conditions for which the assumption that the properties do not depend upon pressure variations will not be satisfied. For boiling systems this limitation has been already recognized and discussed by Christensen and Solberg [56]. In general, it can be expected that the assumption will be satisfied in the low frequency range, i.e., in "chugging" oscillations. When the effects of pressure variations can be neglected then one can use the formulation put forward in [50] and carried out in [51] for boiling systems at subcritical pressures.

The method of solution used in this paper is as follows. A small perturbation is imparted to the inlet velocity. The velocity of the fluid is determined then by integrating the divergence of the velocity. With the

*Indeed, in the critical region some authors reported values approaching a zero velocity. At present neither the exact value of the velocity of sound at the critical point is available nor a satisfactory understanding of the phenomenon has been attained.

velocity known the energy equation is integrated to obtain the time lag τ_b as well as the rate of propagation of enthalpy disturbances. From the enthalpy and from the equation of state we then obtain the density of the medium. The differentiation between the nearcritical region and the supercritical region is achieved by assigning the appropriate expression to the equation of state. With the velocity and the density known, the momentum equation can be integrated. Since the inlet disturbance is small, the momentum equation is first linearized and then integrated to give the characteristic equation.

Because of the linearization of the momentum equation the analysis will be applicable only to cases where the effects of the instability are not so strong to produce large amplitude oscillations. It can be used therefore to predict the conditions of incipient instability, i.e., to determine stability limits. As discussed in [48] and [50, 51] linear effects and formulations have been successful in predicting certain type of instabilities ("chugging" instabilities) in combustion and in boiling systems respectively. A similar result could be expected, therefore, with the present formulation if it is used to predict the onset of "chugging" instabilities at supercritical pressures.

II.6 The Characteristic Equation and Its Applications

The characteristic equation for this problem is an exponential polynomial, it is therefore of the same form as the characteristic equation for combustion instabilities [48], thus

$$f(s) = L_1(s) - e^{-s\tau_b} L_2(s) = 0 \qquad \text{II-1}$$

where L_1 and L_2 are polynomials with coefficients independent of the time and where s is a root of the characteristic equation.

In general, the root s is a complex number; the real part gives the amplification coefficient of the particular oscillatory mode, whereas the imaginary part represents the angular frequency. Since the original perturbation is assumed to be of the form exp[st], a given oscillatory mode will be stable, neutral or unstable depending upon whether the real part of s is less, equal or greater than zero. A sufficient condition for the system to be stable is therefore that the characteristic equation (Eq. II-1) has no roots in the right half of the complex s plane.

Let us examine now what information can be obtained from the characteristic equation as well as the type of practical problems where this information can be applied most usefully. Two such problems were discussed by Crocco and Cheng [48] in connection with the stability analysis of combustion systems. The same discussion can be applied to the present problem.

In the first class of practical problems one is interested in determining whether a <u>given</u> system with specified characteristics, i.e., with specified numerical coefficients is stable or unstable. This is most often a situation that arises during the planning period; i.e., before the system is designed and tested. The characteristic equation can be used to provide an answer to this type of problem. In particular, since the numerical coefficients in the characteristic equation are known, Crocco and Cheng [48] note that the use of Satche's [58] diagram is most useful for analyzing the exponential polinomial obtaining thereby a solution for this type of problem.

In the second class of practical problems one is interested in the qualitative trends of the stability behaviour of a system when various parameters are changed. This is most often a situation that arises during the design period because of the designer's need either to design a system with sufficient safety margins or to modify a given unstable system in order to make it stable. For this kind of problem Crocco and Cheng [48] note that it is advantageous to use the characteristic equation for determining the stability boundary of a certain system. On such a boundary, expressed in terms of the operating characteristics of the system and of the process, the oscillatory mode in question is neither stable nor unstable, i.e., the real part of s vanishes for that mode. The stability boundary divides therefore the space formed by the parameters of a given system into different domains in which the system is stable on one side of the boundary and unstable on the other. If by varying one parameter of the system the stability boundary is shifted in such a way as to decrease the unstable domain, the variation of the parameter has a stabilizing effect and vice versa.

Following the standard procedure the stability boundary is obtained from the characteristic equation by setting $s = i\omega$, where ω is the frequency of the neutral oscillation. Upon separating the real and imaginary parts of the characteristic equation one can eliminate the frequency ω, the resulting equation represents the stability boundary. Two such boundaries, obtained from characteristic equations given by first and second order exponential polynomials, are shown in subsequent chapters.

II.7 Additional Mechanisms Leading to Unstable Operation

Before proceeding with the formulation of the present problem we shall note and examine briefly additional mechanisms which can induce flow oscillations in the nearcritical and supercritical regions. These mechanisms will be analyzed in more detail in separate publications.

It is instructive to note first a general characteristic of oscillatory systems. A necessary condition for maintaining oscillations is that enough energy is supplied to the system at the proper frequency and phase relation in order to overcome the losses due to various damping effects which are always present in real systems. When the rate of energy supplied is controlled by an external source and is independent of the fluctuations inside the systems, the oscillations will build up when the energy is released at a characteristic frequency giving rise to the resonance phenomenon. However, when the system contains itself an energy source, with a property that the energy release depends upon a fluctuation <u>inside</u> the system, then an accidental small disturbance in the system may interact with the source resulting in oscillations of increasing amplitude. For this to take place it is necessary that the energy from the source be fed to the disturbance during part of the cycle.

It was discussed in preceding sections that the system which is analyzed in this paper has the property that the energy release depends upon fluctuations inside the system. Oscillations of the time lag and of the space lag are examples of such fluctuations. We shall examine now other energy sources and fluctuations which may be present in the system.

It was discussed by Rayleigh [59] that internally driven pressure oscillations can occur in a system consisting of a gas flowing through a heated duct. For such oscillations to be maintained Rayleigh notes that the energy must be added to the gas during the moment of greatest condensation and removed during the rarefaction period. This leads to the Rayleigh's criterion which states that a component of the fluctuating heat addition must be in phase with the pressure wave if oscillations are to be thermally driven.

The same criterion can be used to explain a type of oscillation observed at critical and supercritical pressures as well as in boiling mixtures at subcritical pressures. In both systems the heat transfer coefficient is a strong function of pressure. Thus, pressure oscillations may interact with the heat transfer coefficient inducing oscillations of the latter. If these oscillations are in phase, the system may be thermally driven and become oscillatory.

Another mechanism which may induce oscillations at both subcritical and at supercritical pressures is caused by the large compressibility of some parts of the system. At high pressures this is the section of the system where the properties of the fluid pass through the nearcritical region. At pressures below the critical point, this will be the section of the system where subcooled boiling takes place.

Still another mechanism that can induce oscillations at subcritical pressures is caused by the change of flow regime which can induce large fluctuation of the mixture density. These in turn may induce both oscillations of the flow and of the heat transfer coefficient thus providing the driving force necessary for maintaining the oscillations.

It can be expected that each of the mechanisms may be effective over some range of operating conditions. It can be also expected that the resulting oscillations are characterized by a certain frequency range and by particular wave forms. Indeed, several frequency ranges and wave shapes have been reported and described in the references discussed in Chapter I. The mechanisms just described will be the subject of future investigations; in what follows we shall proceed with an analysis of the "chugging" oscillations.

III. The "Heavy" Fluid Region

III.1 The Governing Equation

For a "two-region" approximation the "heavy" fluid region extends from the entrance of the heated duct to the transition point. Note, that for a system with constant energy input, the location of this point will move along the duct when the inlet velocity and/or the inlet enthalpy vary.

It will be assumed that the fluid in this region is incompressible and that the thermodynamic properties are constant. The problem is formulated by considering the three field equations describing the conservation of mass, momentum and of energy in addition to the constitutive equation of state describing the properties of the fluid.

For a one dimensional formulation, used in this analysis, the continuity, energy and momentum equations are given respectively by:

$$\frac{\partial \rho_f}{\partial t} + u \frac{\partial \rho_f}{\partial z} + \rho_f \frac{\partial u}{\partial z} = 0 \qquad \text{III-1}$$

$$\frac{\partial i}{\partial t} + u \frac{\partial i}{\partial z} = \frac{q\xi}{\rho_f A_c} \qquad \text{III-2}$$

and

$$-\frac{\partial P}{\partial z} = \rho_f \frac{\partial u}{\partial t} + \rho_f u \frac{\partial u}{\partial z} + g\rho_f + \frac{f}{2D} \rho_f u^2 \qquad \text{III-3}$$

where the symbols are defined in the Nomenclature.

The constitutive equation of state is given by:

$$\rho_f = \text{const.} \qquad \text{III-4}$$

Equations III-1, 2, 3 and 4 are four equations which specify the four variables P, ρ, u and i in the "heavy" fluid region. These four equations will be integrated to yield P, ρ, u and i as function of space and of time. These will be then used to determine the time lag and the space lag.

III.2 <u>The Equation of Continuity and the Divergence of the Velocity</u>

In view of the assumption of an incompressible "heavy" fluid the continuity equation reduces to the divergence of the velocity

$$\frac{\partial u}{\partial z} = 0 \qquad \text{III-5}$$

whence upon integration we obtain

$$u = u(t) \qquad \text{III-6}$$

The velocity in the "heavy" fluid region is therefore independent of position, it is a function of time only.

In order to analyze the stability problem we shall assume that a small, time dependent velocity variation δu_1, is superimposed on a steady inlet velocity $\bar{u}_1$, thus:

$$u_1(t) = \bar{u}_1 + \delta u_1 = \bar{u}_1 + \varepsilon e^{st} \qquad \text{III-7}$$

where the bar indicates steady state conditions.

III-3 The Energy Equation

With the fluid velocity given by Eq. III-7, the energy equation becomes:

$$\frac{\partial i}{\partial t} + u(t)\frac{\partial i}{\partial z} = \frac{q\xi}{\rho_f A_c} \qquad \text{III-8}$$

This is a first order partial differential equation whose solution can be obtained by means of characteristics [60, 61]. The general solution of Eq. III-8 is of the form:

$$\varphi_2 = f(\varphi_1) \qquad \text{III-9}$$

where

$$\varphi_1(i,t,z) = C_1 \quad \text{and} \quad \varphi_2(i,z,t) = C_2 \qquad \text{III-10}$$

are solutions of any two independent differential equations which imply the relationships:

$$dt = \frac{dz}{u(t)} = \frac{di}{\frac{q\xi}{\rho_f A_c}} \qquad \text{III-11}$$

For example, by taking alternately the first and the second equation, the first and the third equation we obtain the following set

$$\frac{dz}{dt} = \bar{u}_1 + \varepsilon e^{st} \qquad \text{III-12}$$

and

$$\frac{di}{dt} = \frac{q\xi}{\rho_f A_c} \qquad \text{III-13}$$

In order to solve the problem it is necessary to specify the initial and the boundary conditions. These will be specified by letting a particle enter the duct with enthalpy i_1 at time τ_1, (See Figure II-3) thus,

$$i = i_1 \quad \text{at} \quad z = 0 \quad \text{and} \quad t = \tau_1 \qquad \text{III-14}$$

With this boundary and initial condition, one obtains after integrating Eq. III-12 and III-13 the following relations:

$$z = \bar{u}_1 (t-\tau_1) + \varepsilon e^{st} \left[\frac{1 - e^{-s(t-\tau_1)}}{s} \right] \qquad \text{III-15}$$

and

$$i = i_1 + \frac{q\xi}{\rho_f A_c} (t-\tau_1) \qquad \text{III-16}$$

Upon eliminating the time $t-\tau_1$, between these two equations we obtain an expression for the enthalpy as function of space and time, thus:

$$z = \frac{\bar{u}_1 \rho_f (i-i_1) A_c}{q\xi} + \frac{\varepsilon e^{st}}{s} \left[1 - e^{-s(i-i_1)\frac{\rho_f A_c}{q\xi}} \right] \qquad \text{III-17}$$

The first term on the right-hand side is the steady state term, whereas the second one is the transient which accounts for the perturbation of the inlet velocity.

III.4 The Time Lag and The Space Lag

In Section II-2 the time lag τ_b, was defined as the time required for increasing the enthalpy of the fluid from the inlet value i_1 up to the enthalpy at the transition point i_2 (See Figure II-3). Consider now a fluid which enters the duct at time τ_1 and attains the enthalpy i_2 at time τ_2; it follows then from Eq. III-16 that the time lag is given by:

$$\tau_b = \tau_2 - \tau_1 = (i_2 - i_1)\frac{\rho_f A_c}{q\xi} \qquad \text{III-18}$$

which, in view of Eq. III-13 can be also expressed as:

$$\tau_b = \frac{\Delta i_{21}}{\frac{di}{dt}} = \frac{\Delta i_{21}}{\frac{q\xi}{\rho_f A_c}} \qquad \text{III-19}$$

It can be seen that for a given system and at a given pressure the time lag depends only upon the enthalpy difference ($i_2 - i_1 = \Delta i_{21}$) and the heat flux density.

We shall determine now the space lag. For a "two region" approximation. Figure II-3 indicates that the "heavy" fluid region extends up to the transition point where the bulk fluid enthalpy reaches the value of i_2. Inserting i_2 in Eq. III-17 we obtain the following expression for the space lag:

$$\lambda(t) = \frac{\bar{u}_1 \rho_f \Delta i_{21} A_c}{q\xi} + \frac{\varepsilon e^{st}}{s}\left\{1 - e^{-s\frac{\rho_f \Delta i_{21} A_c}{q\xi}}\right\} \qquad \text{III-20}$$

For steady state operation $\varepsilon = 0$, whence we obtain from Eq. III-20 the steady state <u>space lag</u> $\overline{\Lambda}$, thus:

$$\overline{\Lambda} = \frac{\overline{u}_1 \rho_f \Delta i_{21} A_c}{q \xi} \qquad \text{III-21}$$

In view of Eq. III-18 and III-21, we can also express Eq. III-20 as:

$$\Lambda(t) = \overline{u}_1 \tau_b + \varepsilon e^{st} \left\{ \frac{1 - e^{-s\tau_b}}{s} \right\} \qquad \text{III-22}$$

or

$$\Lambda(t) = \overline{\Lambda} + \delta \Lambda \qquad \text{III-23}$$

Several comments are appropriate.

1) Equations III-18 and III-21 indicate that for steady state, i.e., when $\varepsilon = 0$ the time lag τ_b corresponds to the transit time of a fluid particle through the "heavy" fluid region.

2) Equation III-22 shows that the response of the space lag to a variation of the inlet velocity is delayed by a time period equal to the time lag τ_b.

3) If we interpret the enthalpy i_2 as the enthalpy at saturation and therefore the difference Δi_{21} by the subcooling then Eq. III-22 predicts the location of the boiling boundary, i.e., the location where boiling **starts** in a two-phase mixture at subcritical pressures. Indeed, such an expression was derived previously (in [49, 50, 51] among others, using somewhat different approaches) for analyzing oscillations in boiling mixtures.

The time lag τ_b was called there the "evaporation time constant" [49].

III.5 The Momentum Equation

The momentum equation can be integrated now since the velocity in and the boundaries of the "heavy" fluid region have been determined. With the boundary conditions taken as

$$P = P_1 \quad \text{at} \quad z = 0$$

$$P = P_2 \quad \text{at} \quad z = \lambda(t) \qquad \text{III-24}$$

the integrated momentum equation becomes

$$-\int_{P_1}^{P_2} dP = \rho_f \int_0^{\lambda(t)} \left[\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial z} + g + \frac{f}{2D} u_1^2 \right] dz \qquad \text{III-25}$$

where we have taken into account the assumption that the density in the "heavy" fluid region is constant. In view of Eq. III-5, III-6, III-7, and III-23, the integrated momentum equation yields:

$$P_1 - P_2 = \rho_f \left[\frac{\partial \delta u_1}{\partial t} + g + \frac{f}{2D} (\bar{u}_1 + \delta u_1)^2 \right] (\bar{\lambda} + \delta \lambda) \qquad \text{III-26}$$

Linearizing Eq. III-26 and retaining only the terms with the first power of ε we obtain:

$$P_1 - P_2 = g\rho_f \bar{\lambda} + \frac{f\bar{\lambda}}{2D}\rho_f \bar{u}_1^2 +$$

$$+\left\{\rho_f \bar{\lambda}\right\} s\varepsilon e^{st} + \left\{\frac{f\rho_f\bar{\lambda}}{2D} 2\bar{u}_1\right\}\varepsilon e^{st} + \left\{g\rho_f + \frac{f}{2D}\rho_f \bar{u}_1^2\right\}\frac{\varepsilon e^{st}}{s}\left(1 - e^{-s\tau_6}\right)$$

III-27

We shall consider now the pressure drop across the inlet orifice. Defining by k_i a numerical coefficient that takes into account the effect of the geometry of the restriction and of other losses like vena contracta, etc., we can express the inlet pressure drop across the inlet orifice by:

$$P_o - P_1 = k_i \rho_f u_1^2 \qquad \text{III-28}$$

which, upon linearization can be expressed as:

$$P_1 = P_o - k_i \rho_f \bar{u}_1^2 - k_i \rho_f 2\bar{u}_1 \varepsilon e^{st} \qquad \text{III-29}$$

We define now the steady state values of the pressure drop due to body forces (gravity) by:

$$\overline{\Delta P_{bf}} = g\rho_f \bar{\Lambda} \qquad \text{III-30}$$

due to friction by:

$$\overline{\Delta P_{12}} = \frac{f\bar{\Lambda}}{2D}\rho_f \bar{u}_1^2 \qquad \text{III-31}$$

and due to the inlet orifice by:

$$\overline{\Delta P_{01}} = k_i \rho_f \bar{u}_1^2 \qquad \text{III-32}$$

In view of these three relations and upon substituting Eq. III-29 in Eq. III-27, we can express the pressure drop in the "heavy" fluid region by:

$$P_0 - P_2 = \overline{\Delta P_{01}} + \overline{\Delta P_{12}} + \overline{\Delta P_{bf}} +$$

$$+\left\{\rho_f \bar{\lambda}\right\} \frac{d\,\delta u_1}{dt} + 2\left\{\frac{\partial \overline{\Delta P_{01}}}{\partial \bar{u}_1} + \frac{\partial \overline{\Delta P_{12}}}{\partial \bar{u}_1}\right\} \delta u_1 + \left\{\frac{\partial \overline{\Delta P_{bf}}}{\partial \bar{\lambda}} + \frac{\partial \overline{\Delta P_{12}}}{\partial \bar{\lambda}}\right\} \delta\lambda \qquad \text{III-33}$$

where

$$\delta u_1 = \varepsilon e^{st} \qquad \text{III-34}$$

and

$$\delta\lambda = \frac{\varepsilon e^{st}}{s}\left(1 - e^{-s\tau_b}\right) \qquad \text{III-35}$$

The first line on the right-hand side of Eq. III-33 represents the sum of the steady state pressure drops, whereas the second one accounts for the transient response. In particular, the first term is the inertia of the "heavy" fluid region; the second term are the pressure losses due to variation of inlet velocity whereas the last term shows the effect of the varying space lag. Equation III-35 indicates that this last effect is delayed by time lag τ_b. We shall proceed now with the analysis of the "light" fluid region.

IV The "Light" Fluid Region

IV.1 The Governing Equation

For a "two region" approximation the "light" fluid region extends from the transition point to the exit of the heated duct. The problem is formulated again in terms of three field conservation equations and of a constitutive equation of state. However, in contrast to the "heavy" fluid the density in the "light" fluid region is function of enthalpy and of pressure. It was discussed in Chapter II that for "chugging" oscillations, the effects of pressure variations can be neglected. Consequently, the density will be a function of enthalpy only.

The "light" fluid region is described, therefore, in terms of the continuity equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial z} + \rho \frac{\partial u}{\partial z} = 0 \qquad \text{IV-1}$$

of the energy equation

$$\frac{\partial i}{\partial t} + u \frac{\partial i}{\partial z} = \frac{1}{\rho} \frac{q\xi}{A_c} \qquad \text{IV-2}$$

the momentum equation

$$-\frac{\partial P}{\partial z} = \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial z} + g\rho + \frac{f}{2D} \rho u^2 \qquad \text{IV-3}$$

and the constitutive equation of state

$$\rho = \rho(i) \qquad \text{IV-4}$$

or

$$v = v(i) \qquad \text{IV-5}$$

when expressed in terms of the specific volume v .

Equations IV-1, 2, 3 and 4 are four equations in terms of four variables P, ρ , u and i. They are applicable to the "light" fluid region at supercritical pressures. They can be also applied to the two phase flow region at subcritical pressures if, and only if, the relative velocity between the phases can be neglected.

It is emphasized here that the form of the energy and the form of the momentum equation, as given by Eq. IV-2 and Eq. IV-3, are correct <u>only</u> if the relative velocity between the phases is either zero or its effect is negligibly small. If such is not the case, then both Eq. IV-2, and Eq. IV-3 <u>must be modified</u>.

It was discussed in Section II-5 that at high pressures, say above 0.85 of the reduced pressures, the effect of the relative velocity is so small that it can be neglected. The region of interest to this analysis is the high pressure region. It can be expected, therefore, that, in this investigation, both Eq. IV-2 and Eq. IV-3 can be used to approximate, with sufficient accuracy, the energy and the momentum equation for the two phase mixture in the nearcritical region.

In what follows we shall use, therefore, Eq. IV-1 through 4 to describe both the "light" fluid at supercritical pressures and the two phase mixture in the nearcritical region. The differentiation between the "light" fluid and the two phase mixture will be realized by assigning the appropriate expression to the constitutive equation of state. This will be done in the section that follows.

IV.2 The Equation of State

For the "light" fluid the relation between the specific volume and the enthalpy can be obtained either empirically, i.e., from experimental data or it can be approximated by an equation of state such as the perfect gas or the van der Waals' equation etc. It was noted in Section II.1 that for oxygen the perfect gas equation predicts with sufficient accuracy the relation between the specific volume and the enthalpy. Since this fluid is of primary interest to this investigation, the perfect gas equation will be used as the constitutive equation of state for the "light" fluid at supercritical pressures.

Assuming a constant pressure process we have for a perfect gas the following relations

$$dv = \frac{R}{P}\, dT \qquad \text{IV-6}$$

and

$$di = c_p\, dT \qquad \text{IV-7}$$

whence

$$\left(\frac{dv}{di}\right)_P = \frac{R}{P c_p} \qquad \text{IV-8}$$

For a "two-region" approximation the boundary condition for the "light' fluid region is given by:

$$v = v_f \quad at \quad i = i_2 \qquad \text{IV-9}$$

We obtain then the equation of state for the "light" fluid region by integrating Eq. IV-8 subject to the boundary condition given by Eq. IV-8, thus:

$$v(i) = v_f + \frac{R}{P c_p}(i - i_2) \qquad \text{IV-10}$$

We shall derive now the equation of state for the two phase mixture in the nearcritical region. We recall first that the quality x, of a two phase mixture is defined by:

$$X = \frac{G_g}{G_f + G_g} \qquad \text{IV-11}$$

where G_g and G_f are the mass flow rates of the vapor phase and of the liquid phase respectively. We recall also that the specific volume and the enthalpy of a two-phase mixture are given by:

$$v = (1-X)v_f + X v_g \qquad \text{IV-12}$$

and

$$i = (1-X) i_f + X i_g \qquad \text{IV-13}$$

Where v_f, i_f and v_g, i_g are the specific volume and the enthalpy of the liquid and of the vapor respectively.

We obtain then the equation of state for the two phase mixture by eliminating the quality x, between Eq. IV-12 and Eq. IV-13, thus:

$$v(i) = v_f + \frac{\Delta v_{fg}}{\Delta i_{fg}} (i - i_2) \qquad \text{IV-14}$$

Where $\Delta v_{fg} = v_g - v_f$, and where Δi_{fg} is the latent heat of vaporization. Differentiating Eq. IV-14 we obtain for the two phase mixture:

$$\left(\frac{dv}{di}\right)_p = \frac{\Delta v_{fg}}{\Delta i_{fg}} \qquad \text{IV-15}$$

which can be compared to Eq. IV-8 applicable to the "light" fluid at supercritical pressures.

It is important to note that both, the equations of state for the "light" fluid at supercritical pressures, i.e., Eq. IV-10, and the equation of state for a two-phase mixture at subcritical pressures, i.e., Eq. IV-14, are of the same form, i.e., both can be expressed as:

$$v(i) = v_f + \left(\frac{dv}{di}\right)_p (i - i_2) \qquad \text{IV-16}$$

We can use, therefore, Eq. IV-16 for the equation of state in the near-critical as well as in the supercritical region. We shall distinguish one region from the other by substituting either Eq. IV-15 or Eq. IV-8 in Eq. IV-16.

IV.3 The Equation of Continuity and the Divergence of Velocity

Several methods are available [49, 50, 51, 62] for determining the velocity in a boiling mixture. Any of these could be modified and used to determine the velocity in the "light" fluid region. In what follows we shall use the method of Bouré [51].

As in the "heavy" fluid region we shall determine the velocity by integrating the divergence. However, in contrast to the "heavy" fluid region where the divergence is given by Eq. III-5, the divergence in the "light" fluid region is not zero but is obtained from Eq. IV-1, thus:

$$\frac{\partial u}{\partial z} = -\frac{1}{\rho}\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial z}\right] \qquad \text{IV-17}$$

In order to integrate the divergence it is necessary to evaluate the right-hand side of Eq. IV-17. Following Bouré this can be done by means of the energy equation.

Since the density is function of enthalpy only one can write

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial z} = \frac{d\rho}{di}\left[\frac{\partial i}{\partial t} + u\frac{\partial i}{\partial z}\right] \qquad \text{IV-18}$$

Substituting Eq. IV-2 in Eq. IV-18, it follows that:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial z} = \frac{1}{\rho}\,\frac{d\rho}{di}\,\frac{q\xi}{A_c} \qquad \text{IV-19}$$

whence from Eq. IV-19 and IV-17 we can express the divergence as:

$$\frac{\partial u}{\partial z} = -\frac{1}{\rho^2}\frac{d\rho}{di}\frac{q\xi}{A_c} \qquad \text{IV-20}$$

We shall define now the reaction frequency* Ω by:

$$\Omega = \left(\frac{dv}{di}\right)_p \frac{q\xi}{A_c} = \left[-\frac{1}{\rho^2}\frac{d\rho}{di}\right]_p \frac{q\xi}{A_c} \qquad \text{IV-21}$$

It follows then from Eq. IV-8 and Eq. IV-21 that the reaction frequency for the "light" fluid in the supercritical region is given by:

$$\Omega = \frac{R}{PC_p}\frac{q\xi}{A_c} \qquad \text{IV-22}$$

whereas it follows from Eq. IV-15 and Eq. IV-21 that for boiling at subcritical pressures the reaction frequency is given by:

$$\Omega = \frac{\Delta v_{fg}}{\Delta i_{fg}}\frac{q\xi}{A_c} \qquad \text{IV-23}$$

With the reaction frequency defined by Eq. IV-21, it follows then from Eq. IV-20 that we can express the divergence as:

$$\frac{\partial u}{\partial z} = \Omega \qquad \text{IV-24}$$

*The reasons for using this definition are discussed in Section IV-9

The physical significance of this equation is simple: the divergence of the velocity in the "light" fluid region is equal to the volumetric rate of formation of the "light" fluid per unit volume of space.

In order to integrate Eq. IV-24 it is necessary to specify the boundary conditions; these are given by considering the velocity in the "heavy" fluid region, i.e., Eq. III-7. The boundary condition for Eq. IV-24 is therefore:

$$u = \bar{u}_1 + \delta u_1 = \bar{u}_1 + \varepsilon e^{st} \quad \text{at} \quad z = \lambda(t) \tag{IV-25}$$

whence upon integration of Eq. IV-24, we obtain for the velocity in the "light" fluid region the following expression:

$$u(z,t) = \bar{u}_1 + \varepsilon e^{st} + \Omega\left[z - \lambda(t)\right] \tag{IV-26}$$

We note that Eq. IV-26 with Ω given by Eq. IV-23 is the velocity in the two phase boiling mixture, as such it was used already in (49,50,51and 62.)

It is instructive to examine further Eq. IV-26, which, by means of Eq. III-23, can be expressed as:

$$u(z,t) = \bar{u}_1 + \Omega(z - \bar{\lambda}) + \varepsilon e^{st} - \Omega \frac{\varepsilon e^{st}}{s}\left(1 - e^{s\tau_b}\right) \tag{IV-27}$$

We obtain the steady state velocity in the "light" fluid region by letting $\varepsilon = 0$ in Eq. IV-27, thus:

$$\bar{u}_g(z) = \bar{u}_1 + \Omega(z-\bar{\Lambda}) \qquad \text{IV-28}$$

We can rewrite then Eq. IV-27 as:

$$u(z,t) = \bar{u}_g(z) + \delta u_g(t) \qquad \text{IV-29}$$

where the time dependent perturbation of the "light" fluid is given by:

$$\delta u_g(t) = \delta u_1 - \Omega\,\delta\Lambda = \varepsilon e^{st} - \Omega \frac{\varepsilon e^{st}}{s}\left(1 - e^{-s\tau_b}\right) \qquad \text{IV-30}$$

It can be seen from Eq. IV-29 and IV-30 that the flow in the "light" fluid region is affected by both the inlet perturbation as well as the perturbation of the space lag. This last perturbation is delayed by the time lag τ_b (see Figure II-4).

We shall define now several steady state relations which shall be used in following chapters.

By letting z = ℓ in Eq. IV-28 we obtain the <u>steady state velocity at the exit of the heated</u> duct, thus:

$$\bar{u}_g(\ell) = \bar{u}_3 = \bar{u}_1 + \Omega(\ell - \bar{\lambda}) \qquad \text{IV-31}$$

The lengthwise <u>average velocity</u> in the "light" fluid region is defined by:

$$\langle u_g \rangle = \frac{1}{l-\bar{\lambda}} \int_0^{l-\bar{\lambda}} \bar{u}_g(z)\, dz \qquad \text{IV-32}$$

whence from Eq. IV-28 we obtain:

$$\langle u_g \rangle = \bar{u}_1 + \frac{\Omega(l-\bar{\lambda})}{2} \qquad \text{IV-33}$$

From Eq. IV-31 we have:

$$\Omega(l-\bar{\lambda}) = \bar{u}_3 - \bar{u}_1 \qquad \text{IV-34}$$

Substituting this relation in Eq. IV-33 we have the following expressions for the average velocity:

$$\langle u_g \rangle = \bar{u}_1 + \frac{\Omega(l-\bar{\lambda})}{2} = \frac{\bar{u}_3 + \bar{u}_1}{2} \qquad \text{IV-35}$$

The <u>log mean velocity</u> in the "light" fluid region is defined by:

$$u_{lm} = \frac{\bar{u}_3 - \bar{u}_1}{\ln \frac{\bar{u}_3}{\bar{u}_1}} = \frac{\Omega(l-\bar{\lambda})}{\ln \frac{\bar{u}_3}{\bar{u}_1}} \qquad \text{IV-36}$$

where we have taken into account Eq. IV-34.

A fourth relation of interest to this investigation can be obtained from the conservation of momentum G, and the definition of the log mean density. If we denote by ρ_3, the density of the fluid at the exit from the heated duct, then the log mean density in the "light" fluid region is given by:

$$\rho_{lm} = \frac{\rho_3 - \rho_f}{\ln \frac{\rho_3}{\rho_f}} \qquad \text{IV-37}$$

The mean velocity, based on the log mean density, is then obtained by considering the mass flux density, i.e., the momentum G, thus:

$$u_m = \frac{G}{\rho_{lm}} = \frac{\rho_f \bar{u}_1}{\rho_{lm}} \qquad \text{IV-38}$$

which, in view of the preceding relations can be expressed also as:

$$u_m = \frac{G}{\rho_{lm}} = \frac{\bar{u}_3 \bar{u}_1}{\bar{u}_3 - \bar{u}_1} \ln \frac{\bar{u}_3}{\bar{u}_1} = \frac{\bar{u}_3 \bar{u}_1}{u_{lm}} \qquad \text{IV-39}$$

We can proceed now with the solution of the energy equation.

IV-4 The Energy Equation

We can solve the energy equation now since the velocity and the equation of state in the "light" fluid region are specified. By substituting Eq. IV-26 and IV-16 in Eq. IV-2 we can express the energy equation as:

$$\frac{\partial i}{\partial t} + \left[\bar{u}_g(z) + \delta u_g(t)\right]\frac{\partial i}{\partial z} = \left[\frac{1}{\rho_f} + \left(\frac{dv}{di}\right)_p (i-i_2)\right]\frac{q\xi}{A_c} \qquad \text{IV-40}$$

Taking the enthalpy i_2 at the transition point for reference and in view of the definition of the reaction frequency Ω, given by Eq. IV-21, we can rewrite Eq. IV-36 as:

$$\frac{\partial(i-i_2)}{\partial t} + \left[\bar{u}_g(z) + \delta u_g(t)\right]\frac{\partial(i-i_2)}{\partial z} = \frac{q\xi}{\rho_f A_c} + \Omega(i-i_2) \qquad \text{IV-41}$$

The initial and boundary conditions for the energy equation are determined by considering the conditions at the transition point (See Figure II-3); they are given therefore by:

$$i-i_2 = 0 \quad at \quad t = \tau_2 \qquad \text{IV-42}$$

$$i-i_2 = 0 \quad at \quad z = \lambda(\tau_2) = \bar{\lambda} + \varepsilon e^{s\tau_2}\left(\frac{1-e^{-s\tau_b}}{s}\right) \qquad \text{IV-43}$$

Equation IV-41 is again a linear partial differential equation, it can be solved therefore by the method used in solving the energy equation for the "heavy" fluid (See Section III-3). Following this procedure, we obtain in place of Eq. III-11 the following set:

$$dt = \frac{d\zeta}{\bar{u}_g(\zeta) + \delta u_g(t)} = \frac{d(i-i_2)}{\frac{q\xi}{\rho_f A_c} + \Omega(i-i_2)} \qquad \text{IV-44}$$

whence:

$$\frac{d\zeta}{dt} = \bar{u}_g(\zeta) + \delta u_g(t) = \bar{u}_1 + \Omega(\zeta-\bar{\lambda}) + \varepsilon e^{st}\,\frac{s-\Omega+\Omega e^{-s\tau_b}}{s} \qquad \text{IV-45}$$

and

$$\frac{d(i-i_2)}{dt} = \frac{q\xi}{\rho_f A_c} + \Omega(i-i_2) \qquad \text{IV-46}$$

Integrating first Eq. IV-46 and taking into account the initial conditions given by Eq. IV-42 we obtain:

$$\ln\left[1 + \Omega(i-i_2)\frac{\rho_f A_c}{q\xi}\right] = \Omega(t-\tau_2) \qquad \text{IV-47}$$

whence:

$$i - i_2 = \frac{q\xi}{\Omega\rho_f A_c}\left[e^{\Omega(t-\tau_2)} - 1\right] \qquad \text{IV-48}$$

In order to integrate Eq. IV-45 we note that it is of the form of:

$$\frac{dz}{dt} - \Omega z = h(t) \qquad \text{IV-49}$$

whose solution is:

$$pz - \int p(\xi)\, h(\xi)\, d\xi = \text{Const.} \qquad \text{IV-50}$$

where $p = e^{-\int \Omega\, d\xi} = e^{-\Omega t}$

The integral of Eq. IV-45 is therefore:

$$z e^{-\Omega t} + \left[\frac{\bar{u}_1}{\Omega} - \bar{\Lambda} \right] e^{-\Omega t} - \frac{\varepsilon e^{(s-\Omega)t}}{s-\Omega} \left(\frac{s - \Omega + \Omega e^{-s\tau_b}}{s} \right) = C_1 \qquad \text{IV-51}$$

which, in view of Eq. IV-48, can be expressed as:

$$\frac{e^{-\Omega \tau_2}}{1 + \frac{\Omega \rho_f A_c}{q \xi}(i - i_2)} \left[z + \frac{\bar{u}_1}{\Omega} - \bar{\Lambda} - \varepsilon e^{st} \frac{s - \Omega + \Omega e^{-s\tau_b}}{s(s-\Omega)} \right] = C_1 \qquad \text{IV-52}$$

The value of the constant of integration is evaluated by means of Eq. IV-42 and IV-43, thus:

$$C_1 = e^{-\Omega \tau_2} \left[\frac{\bar{u}_1}{\Omega} + \varepsilon e^{s\tau_2} \left(\frac{1 - e^{-s\tau_b}}{s} \right) - \varepsilon e^{s\tau_2} \frac{s - \Omega + \Omega e^{-s\tau_b}}{s(s-\Omega)} \right] \qquad \text{IV-53}$$

Substituting Eq. IV-53 in Eq. IV-51 we obtain after some re-arrangement:

$$e^{-\Omega t}\left\{\bar{u}_1+\Omega(z-\bar{\lambda}) - \frac{\Omega}{s-\Omega}\,\varepsilon e^{st}\left[1-\Omega\left(\frac{1-e^{-s\tau_b}}{s}\right)\right]\right\} =$$

$$e^{-\Omega\tau_2}\left\{\bar{u}_1 + \Omega e^{-s(t-\tau_2)}\varepsilon e^{st}\left(\frac{1-e^{-s\tau_b}}{s}\right) - \frac{\Omega}{s-\Omega}e^{-s(t-\tau_2)}\varepsilon e^{st}\left[1-\Omega\left(\frac{1-e^{-s\tau_b}}{s}\right)\right]\right\} \tag{IV-54}$$

which, in view of Eq. III-23 and Eq. IV-30, can be expressed as

$$e^{-\Omega t}\left\{\bar{u}_g(z) - \frac{\Omega}{s-\Omega}\,\delta u_g\right\} =$$

$$e^{-\Omega\tau_2}\left\{\bar{u}_1 - e^{-s(t-\tau_2)}\Omega\,\delta\lambda - \frac{\Omega}{s-\Omega}e^{-s(t-\tau_2)}\delta u_g\right\} \tag{IV-55}$$

By substituting Eq. IV-48 in Eq. IV-55 we obtain the solution of the energy equation for the specified initial and boundary conditions, thus

$$1 + \frac{\Omega\rho_f A_c(i-i_2)}{q\xi} = \frac{\bar{u}_g(z) - \frac{\Omega}{s-\Omega}\delta u_g}{\bar{u}_1 + e^{-s(t-\tau_2)}\Omega\,\delta\lambda - \frac{\Omega}{s-\Omega}e^{-s(t-\tau_2)}\delta u_g} \tag{IV-56}$$

from which we obtain the enthalpy for the "light" fluid region

$$i - i_2 = \frac{q\xi}{\rho_f A_c \Omega}\left\{\frac{\overline{u}_g(z) - \frac{\Omega}{s-\Omega}\,\delta u_g}{\overline{u}_1 + e^{-s(t-\tau_2)}\,\Omega\,\delta\lambda - \frac{\Omega}{s-\Omega}\,e^{-s(t-\tau_2)}\,\delta u_g} - 1\right\} \qquad \text{IV-57}$$

Expanding and retaining only the first power of ε we obtain after some rearrangement

$$i - i_2 = \frac{q\xi(z-\overline{\lambda})}{\overline{u}_1 \rho_f A_c} - \frac{q\xi}{\overline{u}_1 \rho_f A_c}\,\frac{1}{s-\Omega}\left\{\delta u_g - \frac{\overline{u}_g(z)}{\overline{u}_1}\,e^{-s(t-\tau_2)} \cdot e^{-s\tau_6}\,\delta u_1\right\} \qquad \text{IV-58}$$

If we let the perturbation go to zero, i.e., $\varepsilon = 0$, we obtain from Eq. IV-58 the enthalpy for steady state operation, thus

$$i - i_2 = \frac{q\xi(z-\overline{\lambda})}{\overline{u}_1 \rho_f A_c} \qquad \text{IV-59}$$

IV-5 The Residence Time

It is of interest to evaluate now the steady state residence, i.e., the transit time of a particle in the heated duct. Denoting by i_3 the enthalpy at the exit and by τ_3 the time when a particle reaches this enthalpy we obtain from Eq. IV-47 the residence time in the "light" fluid region, thus:

$$\tau_3-\tau_2 = \frac{1}{\Omega} \ln\left\{1 + \Omega(i_3-i_2)\frac{\rho_f A_c}{q\xi}\right\}$$ IV-60

which, in view of Eq. IV-21, can be expressed also as:

$$\tau_3-\tau_2 = \left(\frac{di}{dv}\right)_p \frac{A_c}{q\xi} \ln\left\{1+\left(\frac{di}{dv}\right)_p \rho_f \Delta i_{32}\right\}$$ IV-61

Denoting by $\dot{Q} = q\xi l$, the rate of energy transfer to the entire duct and by W, the mass flow rate, we can express the total energy balance in steady state as:

$$(i_3-i_2)+(i_2-i_1) = \frac{\dot{Q}}{W}$$ IV-62

Substituting this relation in Eq. IV-61 and in view of Eq. III-18 we obtain the following expression for the residence time in the heated duct:

$$\tau_3-\tau_1 = \frac{\Delta i_{21}\rho_f A_c}{q\xi} + \left(\frac{di}{dv}\right)_p \frac{A_c}{q\xi} \ln\left\{1+\left(\frac{dv}{di}\right)_p \frac{\rho_f \dot{Q}}{W}\left(1-\frac{W\Delta i_{21}}{\dot{Q}}\right)\right\}$$ IV-63

IV-6 The Density and the Density Perturbation

The density in the "light" fluid region is given by the equation of state, i.e., by Eq. IV-16, which, in view of the definition of the reaction frequency Ω, given by Eq. IV-21, can be expressed also as:

$$\frac{\rho_f}{\rho(i)} = 1 + \left(\frac{dv}{di}\right)_P \rho_f (i - i_2) = 1 + \Omega \frac{\rho_f A_c}{q \xi}(i - i_2) \qquad \text{IV-64}$$

Since the enthalpy in the "light" fluid region is given by the solution of the energy equation, i.e., by Eq. IV-56 we can express the density as function of time or as function of time and space. Thus by substituting Eq. IV-64 in Eq. IV-47 we obtain:

$$\frac{\rho(t,\tau_2)}{\rho_f} = e^{-\Omega(t-\tau_2)} \qquad \text{IV-65}$$

whereas by substituting Eq. IV-64 in Eq. IV-56 we obtain:

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1 + e^{-s(t-\tau_2)}\Omega\delta\lambda - \frac{\Omega}{s-\Omega} e^{-s(t-\tau_2)}\delta u_g}{\bar{u}_g(z) - \frac{\Omega}{s-\Omega}\delta u_g} \qquad \text{IV-66}$$

which, by means of Eq. IV-30, III-20 and III-18, can be expressed also as

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1 - \frac{\Omega}{s-\Omega} e^{-s(t-\tau_2)}\delta u_1}{\bar{u}_g(z) - \frac{\Omega}{s-\Omega}\delta u_g} \qquad \text{IV-67}$$

or, in view of Eq. IV-65, it can be transformed in

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1 - \frac{\Omega}{s-\Omega}\left[\frac{\rho(z,t)}{\rho_f}\right]^{s/\Omega} e^{-s\tau_b}\,\delta u_1}{\bar{u}_g(z) - \frac{\Omega}{s-\Omega}\,\delta u_g} \qquad \text{IV-68}$$

Using again the binomial expansion and retaining only the first power of ε we can express Eq. IV-67 as:

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1}{\bar{u}_g(z)} + \frac{\Omega}{s-\Omega}\,\frac{\bar{u}_1}{\bar{u}_g(z)}\left\{\frac{\delta u_g}{\bar{u}_g(z)} - e^{-s(t-\tau_1)}\,\frac{\delta u_1}{\bar{u}_1}\right\} \qquad \text{IV-69}$$

Similarly, we can expand Eq. IV-68 and express it as:

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1}{\bar{u}_g(z)} + \frac{\Omega}{s-\Omega}\,\frac{\bar{u}_1}{\bar{u}_g(z)}\left\{\frac{\delta u_g}{\bar{u}_g(z)} - \left[\frac{\bar{u}_1}{\bar{u}_g(z)}\right]^{s/\Omega} e^{-s\tau_b}\,\frac{\delta u_1}{\bar{u}_1}\right\} \qquad \text{IV-70}$$

where the steady state velocity $\bar{U}_g(z)$ is given by Eq. IV-28.

By letting $\varepsilon = 0$ in Eq. IV-66 we can obtain the local steady state density in the "light" fluid region, thus:

$$\frac{\bar{\rho}(z)}{\rho_f} = \frac{\bar{u}_1}{\bar{u}_g(z)} = \frac{\bar{u}_1}{\bar{u} + \Omega(z-\bar{\lambda})} \qquad \text{IV-71}$$

We shall define now several steady state relations which will be used in the following sections.

By letting $z = \ell$ in Eq. IV-71 we obtain the density at the exit from the heated duct thus:

$$\rho(\ell) = \rho_3 = \frac{\rho_1 \bar{u}_1}{\bar{u}_1 + \Omega(\ell - \bar{\lambda})} = \frac{G}{\bar{u}_3} \qquad \text{IV-72}$$

where we have taken into account Eq. IV-31.

We shall define now the average density in the "light" fluid region by:

$$\langle \rho_q \rangle = \frac{1}{\ell - \bar{\lambda}} \int_0^{\ell - \bar{\lambda}} \bar{\rho}(z)\, dz \qquad \text{IV-73}$$

whence from Eq. IV-71 we obtain:

$$\langle \rho_q \rangle = \frac{\rho_1 \bar{u}_1}{\Omega(\ell - \bar{\lambda})} \ln \frac{\bar{u}_3}{\bar{u}_1} \qquad \text{IV-74}$$

In view of the definition of the log mean velocity $u_{\ell m}$ given by Eq. IV-36 this average density can be expressed as:

$$\langle \rho_g \rangle = \frac{G}{\bar{u}_3 - \bar{u}_1} \ln \frac{\bar{u}_3}{\bar{u}_1} = \frac{G}{u_{\ell m}} \qquad \text{IV-75}$$

We have already defined the log mean density by:

$$\rho_{lm} = \frac{\rho_f - \rho_3}{\ln \frac{\rho_f}{\rho_3}} \qquad \text{IV-76}$$

where ρ_3 is given by Eq. IV-72.

A fourth expression can be obtained from the definition of the average velocity $\langle u_g \rangle$ given by Eq. IV-35 and the momentum G . We can express therefore a mean density, based on the average velocity, by:

$$\rho_m = \frac{G}{\langle u_g \rangle} = \frac{G}{\bar{u}_1 + \frac{\Omega(\ell - \lambda)}{2}} = \frac{2G}{\bar{u}_3 + \bar{u}_1} \qquad \text{IV-77}$$

With the steady state density in the "light" fluid vapor given by Eq. IV-71, we can express Eq. IV-69 as

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{\rho}(z)}{\rho_f} + \frac{\delta\rho(z,t)}{\rho_f} \qquad \text{IV-78}$$

where the density perturbation is given by

$$\delta\rho(z,t) = \frac{\Omega}{s-\Omega}\,\frac{G}{\bar{u}_g(z)}\left\{\frac{\delta u_g}{\bar{u}_g(z)} - e^{-s(t-\tau_1)}\,\frac{\delta u_1}{\bar{u}_1}\right\} \qquad \text{IV-79}$$

which, in view of Eq. IV-30, can be expressed also as:

$$\delta\rho(z,t) = \frac{\Omega}{s-\Omega}\,\frac{G}{\bar{u}_g(z)}\left\{\frac{\delta u_1}{\bar{u}_g(z)} - \frac{\Omega\,\delta\lambda}{\bar{u}_g(z)} - e^{-s(t-\tau_1)}\,\frac{\delta u_1}{\bar{u}_1}\right\} \qquad \text{IV-80}$$

By letting $z = \ell$ in Eq. IV-79 we obtain the density perturbation at the exit from the heated duct, thus

$$\delta\rho_3 = \frac{\Omega}{s-\Omega}\,\frac{G}{\bar{u}_3}\left\{\frac{\delta u_g}{\bar{u}_3} - e^{-s(\tau_3-\tau_1)}\,\frac{\delta u_1}{u_1}\right\} \qquad \text{IV-81}$$

It can be seen from the preceeding equations that in the "light" fluid region the density perturbation is affected by <u>both</u> the perturbation of the inlet velocity and by the variation of the space lag. Furthermore, the effect of the inlet velocity perturbation is delayed by a delay time. Equations IV-30 and IV-80 are the quantitative expressions for the flow and density variations in the "light" fluid region which were qualitatively described in Section II-4.

With the density and the velocity in the "light" fluid region given by the expressions derived in this section and in Section IV-3 respectively, we are in the position to integrate the momentum equation.

IV.7 The Momentum Equation

In order to integrate the momentum equation it is necessary to specify the boundary conditions, these are given by:

$$P = P_2 \quad \text{at} \quad z = \lambda(t)$$
$$P = P_3 \quad \text{at} \quad z = l \qquad \text{IV-82}$$

whence the integrated momentum equation becomes:

$$-\int_{P_2}^{P_3} dP = \int_{\lambda(t)}^{l} \left\{ \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial z} + g\rho + \frac{f}{2D} \rho u^2 \right\} dz \qquad \text{IV-83}$$

The expressions for the density and the velocity which should be substituted in this equation are given by Eq. IV-78 and Eq. IV-27, i.e., Eq. IV-29 respectively. We shall consider now each term of Eq. IV-83 separately.

IV.7.1 The Inertia Term

The inertia term in the momentum equation is given by:

$$\Delta P_I = \int_{\lambda(t)}^{l} \rho \frac{\partial u}{\partial t} dz \qquad \text{IV-84}$$

Substituting Eq. IV-78 and Eq. IV-27 in Eq. IV-84 and retaining only the first power in ε we get:

$$\Delta P_I = \frac{\rho_F \bar{u}_1}{\Omega} \ln \frac{\bar{u}_3}{\bar{u}_1} \varepsilon e^{st} (s - \Omega + \Omega e^{-s\tau_b}) \qquad \text{IV-85}$$

In view of the definition of the average density and of the velocity perturbation given by Eq. IV-75 and Eq. IV-30, respectively, the inertia term can be expressed as:

$$\Delta P_I = (l - \bar{\lambda}) \langle \rho_g \rangle \frac{d\,\delta u_g}{dt} \qquad \text{IV-86}$$

IV.7.2 <u>The Convective Acceleration Term</u>

The convective acceleration term in Eq. IV-83 is given by:

$$\Delta P_a = \int_{\lambda(t)}^{l} \rho u \frac{\partial u}{\partial z} dz \qquad \text{IV-87}$$

Substituting Eq. IV-78 and Eq. IV-27 in Eq. IV-87 and retaining only the first power in ξ we obtain upon integration:

$$\Delta P_a = G\Omega(l-\bar{\lambda}) - G\Omega\varepsilon e^{st}\frac{1-e^{-s\tau_b}}{s} +$$

$$G \ln\frac{\bar{u}_3}{\bar{u}_1}\varepsilon e^{st}\frac{s-\Omega+\Omega e^{-s\tau_s}}{s} +$$

$$+ G \ln\frac{\bar{u}_3}{\bar{u}_1}\varepsilon e^{st}\frac{\Omega}{s-\Omega}\frac{s-\Omega+\Omega e^{-s\tau_b}}{s}$$

IV-88

$$- G\varepsilon e^{st}e^{-s\tau_b}\left(\frac{\Omega}{s-\Omega}\right)^2\left\{1 - e^{-s(\tau_3-\tau_2)}\frac{\bar{u}_3}{\bar{u}_1}\right\}$$

It is of interest to examine the physical significa[illegible]ce of the various terms.

If we let $\varepsilon = 0$ in Eq. IV-88, we obtain the steady state acceleration pressure drop $\overline{\Delta P_a}$, which, in view of the definitions given in Sections IV-3 and IV-6, can be expressed as:

$$\overline{\Delta P_a} = G\Omega(l-\bar{\lambda}) = G(\bar{u}_3-\bar{u}_1) = \langle\rho_q\rangle u_{lm}(\bar{u}_3-\bar{u}_1) =$$

$$= \langle\rho_q\rangle u_{lm}^2 \ln\frac{\bar{u}_3}{\bar{u}_1} = G^2\frac{\rho_f-\rho_3}{\rho_f\rho_3} = \rho_{lm}\bar{u}_1\bar{u}_3\ln\frac{\bar{u}_3}{\bar{u}_1} \quad \text{IV-89}$$

$$= \rho_{lm}u_m(\bar{u}_3-\bar{u}_1)$$

The second term in Eq. IV-88 can be expressed by means of the Eq. IV-89 and of the space lag variation defined by Eq. III-22, thus

$$G\Omega\varepsilon e^{st}\left\{\frac{1-e^{-s\tau_b}}{s}\right\} = \frac{\overline{\Delta P_a}}{(l-\lambda)}\,\delta\lambda \qquad \text{IV-90}$$

It shows, therefore, the influence of the variation of the space lag on the acceleration pressure drop in the "light" fluid region.

In view of Eq. IV-89 and Eq. IV-30, the third term in Eq. IV-88 can be expressed as

$$G \ln\frac{\bar{u}_3}{\bar{u}_1}\,\varepsilon e^{st}\,\frac{s-\Omega+\Omega e^{-s\tau_b}}{s} - G\frac{(\bar{u}_3-\bar{u}_1)}{u_{lm}}\,\delta u_g = \frac{\overline{\Delta P_a}}{u_{lm}}\,\delta u_g \qquad \text{IV-91}$$

It expresses, therefore, the influence of the velocity perturbation in the "light" fluid region on the acceleration pressure drop.

The last two terms in Eq. IV-88 stem from the density perturbation term in Eq. IV-70, i.e., from

$$\rho_f\,\frac{\Omega}{s-\Omega}\int_{\Lambda(t)}^{l}\left\{\frac{\bar{u}_1}{\bar{u}_g(z)}\,\frac{\delta u_g}{\bar{u}_g(z)} - \left(\frac{\bar{u}_1}{\bar{u}_g(z)}\right)^{\frac{s}{\Omega}+1} e^{-s\tau_b}\right\}\bar{u}_g(z)\,\frac{\partial \bar{u}_g(z)}{\partial z}\,dz =$$

IV-92

$$= \frac{\Omega}{s-\Omega}\left\{G\ln\frac{\bar{u}_3}{\bar{u}_1}\,\varepsilon e^{st}\,\frac{s-\Omega+\Omega e^{-s\tau_b}}{s} - G\varepsilon e^{st} e^{-s\tau_b}\left(\frac{\Omega}{s-\Omega}\right)\left[1-e^{-s(\tau_3-\tau_1)}\,\frac{\bar{u}_3}{\bar{u}_1}\right]\right\}$$

The third term in Eq. IV-88, i.e., the first term on the right hand side of Eq. IV-92 can be expressed in terms of Eq. IV-89 thus

$$\frac{\Omega}{s-\Omega} G \ln \frac{\bar{u}_3}{\bar{u}_1} \varepsilon e^{st} \frac{s-\Omega+\Omega e^{-s\tau_b}}{s} = \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_a}}{u_{1m}} \delta u_g \qquad \text{IV-93}$$

It shows, therefore, the effect of the variation of the velocity in the "light" fluid region on the density and, therefore, on the acceleration pressure drop in that region.

The physical meaning of the last term on Eq. IV-88, i.e., Eq. IV-92 is not as clear as that of the other terms in Eq. IV-88. An insight can be gained however, by considering the upper and lower limits of the integral

$$I_4 = \rho_f \frac{\Omega}{s-\Omega} \int_{\lambda(t)}^{l} \left\{ \left(\frac{\bar{u}_1}{\bar{u}_{g(z)}} \right)^{\frac{s}{\Omega}+1} e^{-s\tau_b} \frac{\delta u_1}{\bar{u}_{g(z)}} \right\} \bar{u}_{g(z)} \frac{\partial \bar{u}_{g(z)}}{\partial z} dz \qquad \text{IV-94}$$

It is shown in the Appendix C that this integral is bounded by:

$$\frac{\Omega}{s-\Omega} G\Omega(l-\bar{\lambda}) e^{-s(\tau_3-\tau_1)} \frac{\delta u_1}{\bar{u}_1} < I_4 < \frac{\Omega}{s-\Omega} G\Omega(l-\bar{\lambda}) e^{-s\tau_b} \frac{\delta u_1}{\bar{u}_1} \qquad \text{IV-95}$$

which, in view of Eq. IV-89, can be expressed as

$$\frac{\Omega}{s-\Omega}\ \frac{\overline{\Delta P_a}}{\bar{u}_1}\ e^{-s(\tau_3-\tau_1)}\ \delta u_1 < I_4 < \frac{\Omega}{s-\Omega}\ \frac{\overline{\Delta P_a}}{u_1}\ e^{-s\tau_b}\ \delta u_1 \qquad \text{IV-96}$$

We note that a simple expression can be obtained by setting $\bar{u}_1/\bar{u}_q(z) = \bar{u}_1/\bar{u}_3$ in Eq. IV-94, this approximation results in the following expression for the integral I_4:

$$I_4^* = \frac{\Omega}{s-\Omega} G\Omega(1-\bar{\lambda})\, e^{-s(\tau_3-\tau_1)} \frac{\delta u_1}{u_1} =$$

$$= \frac{\Omega}{s-\Omega}\ \frac{\overline{\Delta P_a}}{\bar{u}_1}\ e^{-s(\tau_3-\tau_1)}\ \delta u_1 \qquad \text{IV-97}$$

The physical meaning of the integral I_4 is now clear: it expresses the effect of the perturbation of the inlet velocity on the density (see Eq. IV-69) and, therefore, on the acceleration pressure drop in the "light" fluid region. This effect is delayed by a delay time equal to τ_b or to $(\tau_3 - \tau_1)$ depending on whether we use this upper or lower bound for the integral I_4.

By substituting Eq. IV-89, IV-90, IV-91, IV-93 in Eq. IV-88 and by expressing the integral I_4 in terms of the approximation given by Eq. IV-97 we obtain for the acceleration pressure drop on the "light" fluid region the following expression:

$$\Delta P_a = \overline{\Delta P_a} - \frac{\overline{\Delta P_a}}{(l-\lambda)} \delta\lambda + \frac{\overline{\Delta P_a}}{u_{1m}} \delta u_q +$$

$$+ \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_a}}{u_{1m}} \delta u_q - \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_a}}{u_1} e^{-s(\tau_3-\tau_1)} \delta u_1$$

IV-98

where the steady state acceleration pressure drop $\overline{\Delta P_a}$ is given by Eq. IV-89.

IV.7.3 The Gravitational Term

The gravitational term in the momentum equation is given by

$$\Delta P_{bg} = \int_{\lambda(t)}^{l} g\rho \, dz \qquad \text{IV-99}$$

Substituting Eq. IV-78 and retaining only the first order terms of we obtain after integration:

$$\Delta P_{bg} = g \frac{\rho_f \bar{u}_1}{\Omega} \ln \frac{\bar{u}_3}{\bar{u}_1} - g\rho_f \varepsilon e^{st} \left(\frac{1-e^{-s\tau_6}}{s} \right) +$$

IV-100

$$+ g(l-\bar{\lambda}) \frac{\rho_f}{\bar{u}_3} \varepsilon e^{st} \frac{\Omega}{s-\Omega} \frac{s-\Omega+\Omega e^{-s\tau_5}}{s} - I_4$$

where

$$I_4 = g\rho_f \int_{\Lambda(t)}^{l} \frac{\Omega}{s-\Omega} \left\{ \left(\frac{u_1}{\bar{u}_g(z)} \right)^{s/\Omega} e^{-s\tau_b} \frac{\delta u_1}{\bar{u}_g(z)} \right\} dz =$$

$$= \frac{g\rho_f}{\Omega} \frac{\Omega}{s-\Omega} \frac{\Omega}{s} e^{-s\tau_b} \left[1 - \left(\frac{u_1}{\bar{u}_3} \right)^{s/\Omega} \right] \delta u_1 \qquad \text{IV-101}$$

The physical meaning of the various terms is as follows:

We obtain the steady state gravitational pressure drop by letting $\varepsilon = 0$ in Eq. IV-100, thus in view of the definitions given by Eq. IV-36 and IV-75 we have:

$$\overline{\Delta P}_{bg} = g(l-\bar{\Lambda}) \frac{\rho_f \bar{u}_1}{\Omega(l-\bar{\Lambda})} \ln \frac{\bar{u}_3}{u_1} = g(l-\bar{\Lambda}) \frac{G}{u_{lm}} = \qquad \text{IV-102}$$

$$= g(l-\bar{\Lambda}) \langle \rho_g \rangle$$

The second term in Eq. IV-102 can be expressed by means of Eq. IV-102 and Eq. III-22, thus

$$g\rho_f \varepsilon e^{st} \left(\frac{1-e^{-s\tau_l}}{s} \right) = \frac{g(l-\Lambda)}{(l-\Lambda)} \frac{\rho_f u_1}{u_{lm}} \frac{u_{lm}}{u_1} \delta\Lambda = \qquad \text{IV-103}$$

$$= \frac{\overline{\Delta P}_{bg}}{(l-\bar{\Lambda})} \frac{u_{lm}}{u_1} \delta\Lambda$$

It expresses, therefore, the effect of space lag variation on the gravitational pressure drop.

The third term in Eq. IV-100 can be expressed by means of Eq. IV-39, IV-75 and IV-103, thus

$$g(1-\bar{\lambda})\ \frac{\rho_f}{\bar{u}_3}\ \varepsilon e^{st}\ \frac{\Omega}{s-\Omega}\ \frac{s-\Omega+\Omega e^{-s\tau_b}}{s} = \frac{\Omega}{s-\Omega}\ g(1-\bar{\lambda})\ \frac{G}{\bar{u}_3\bar{u}_1}\ \delta u_g$$

$$= \frac{\overline{\Delta P_{bg}}}{u_m}\ \frac{\Omega}{s-\Omega}\ \delta u_g \qquad \text{IV-104}$$

where the mean velocity u_m is defined by Eq. IV-39. This term then represents the effect of velocity perturbation in the "light" fluid region on the density and therefore on the gravitational pressure drop in this region.

The physical meaning of the last term in Eq. IV-100 can be explained again by expressing the integral I_4 in Eq. IV-101 by its upper and lower bounds (see Appendix C)

$$\frac{\Omega}{s-\Omega}\ g(1-\lambda)\ \frac{G}{\bar{u}_1\bar{u}_3}\ e^{-s(\tau_3-\tau_1)}\ \delta u_1 < I_4 < \frac{\Omega}{s-\Omega}\ g(1-\bar{\lambda})\ \frac{G}{\bar{u}_1\bar{u}_1}\ e^{-s\tau_b}\ \delta u_1 \qquad \text{IV-105}$$

which in view of Eq. IV-39, IV-75 and IV-103, can be expressed as

$$\frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_{bg}}}{u_m} e^{-s(\tau_3-\tau_1)} \delta u_1 < I_4 < \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_{bg}}}{\bar{u}_1} \frac{u_{lm}}{\bar{u}_1} e^{-s\tau_b} \delta u_1$$

IV-106

A simple expression for the integral I_4 can be obtained by using the same approximation which was used in deriving Eq. IV-97. Thus, if we let $\bar{u}_1/\bar{u}_g(z) = \bar{u}_1/\bar{u}_3$ in Eq. IV-101 we obtain after integration the following approximation

$$I_4^* = \frac{\Omega}{s-\Omega} \frac{g\rho_f}{\Omega} \ln \frac{\bar{u}_3}{\bar{u}_1} e^{-s(\tau_3-\tau_1)} \delta u_1 =$$

$$= \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_{bg}}}{\bar{u}_1} e^{-s(\tau_3-\tau_1)} \delta u_1$$

IV-107

By comparing Eq. IV-107 with Eq. IV-106 it can be seen that I_4^* has a value which falls between the two bounds given by Eq. IV-106. The last term in Eq. IV-100 expresses therefore, the effect of the inlet velocity perturbation on one density (see Eq. IV-79) and on the gravitational pressure drop in the "light" fluid region. Furthermore, this effect is delayed by a time delay equal to τ_b or $\tau_3-\tau_1$ depending on whether we use the upper or lower bound for the integral I_4.

By substituting Eq. IV-102, IV-103, IV-104 in Eq. IV-100 and by expressing the integral I_4 in terms of the intermediate approximation I_4^* given by Eq. IV-107 we obtain for the gravitational pressure drop in the "light" fluid region the following expression:

$$\Delta P_{bg} = \overline{\Delta P}_{bg} - \frac{\overline{\Delta P}_{bg}}{1-\bar{\lambda}} \frac{u_{in}}{u_1} \delta\Lambda +$$

$$+ \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P}_{bg}}{u_m} \delta u_g - \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P}_{bg}}{\bar{u}_1} e^{-s(\tau_3-\tau_1)} \delta u_1 \qquad \text{IV-108}$$

where the steady state gravitational presure drop is given by Eq. IV-102.

IV.7.4 The Frictional Pressure Drop

The frictional term in the momentum equation is given by

$$\Delta P_{23} = \int_{\lambda(t)}^{L} \frac{f}{2D} \rho u^2 \, dz \qquad \text{IV-109}$$

Substcituting Eq. IV-78 and IV-27 in Eq. IV-109 and retaining only the first order terms in ε we obtain after integration the following expression

$$\Delta P_{23} = \frac{f(1-\bar{\lambda})}{2D} \rho_f \bar{u}_1 \left[\bar{u}_1 + \frac{\Omega(1-\bar{\lambda})}{2} \right] - \frac{f}{2D} \rho_1 \bar{u}_1^2 \varepsilon e^{st} \left(\frac{1-e^{-s\tau_b}}{s} \right)$$

$$+ 2 \frac{f(1-\bar{\lambda})}{2D} \rho_f \bar{u}_1 \varepsilon e^{st} \frac{s-\Omega+\Omega e^{-s\tau_b}}{s} + \qquad \text{IV-110}$$

$$+ \frac{\Omega}{s-\Omega} \frac{f(1-\bar{\lambda})}{2D} \rho_f \bar{u}_1 \varepsilon e^{st} \frac{s-\Omega+\Omega e^{-s\tau_b}}{s} - I_5$$

where the integral I_5 is given by

$$I_5 = \frac{f\rho_f}{2D}\int_{\Lambda(t)}^{L} \frac{\Omega}{s-\Omega}\left\{\left[\frac{\bar{u}_1}{\bar{u}_g(z)}\right]^{s/\Omega} e^{-s\tau_b}\frac{\delta u_1}{\bar{u}_g(z)}\right\}\bar{u}_g(z)^2\,dz = \qquad \text{IV-111}$$

$$= \frac{f}{2D}G\bar{u}_1\frac{\Omega}{s-\Omega}\frac{e^{-s\tau_b}}{\Omega}\frac{\Omega}{s-2\Omega}\left\{1-\left(\frac{\bar{u}_1}{\bar{u}_g(z)}\right)^{\frac{s}{\Omega}-2}\right\}$$

The physical meaning of the various terms is as follows.

We obtain the steady state frictional pressure drop by letting $\varepsilon = 0$ in Eq. IV-110, thus in view of Eq. IV-35 we can write

$$\overline{\Delta P}_{23} = \frac{f(1-\bar{\Lambda})}{2D}\rho_1\bar{u}_1\left[\bar{u}_1 + \frac{\Omega(1-\bar{\Lambda})}{2}\right] = \qquad \text{IV-112}$$

$$= \frac{f(1-\bar{\Lambda})}{2D}G\langle u_g\rangle$$

The second term in Eq. IV-110 can be expressed by means of Eq. IV-112 and Eq. III-22 thus:

$$\frac{f}{2D}\rho_f\bar{u}_1^2\varepsilon e^{st}\left(\frac{1-e^{-s\tau_b}}{s}\right) = \frac{\overline{\Delta P}_{23}}{(1-\bar{\Lambda})}\frac{\bar{u}_1}{\langle u_g\rangle}\delta\lambda \qquad \text{IV-113}$$

It expresses therefore the effect of the space lag variation on the frictional pressure drop.

The third term in Eq. Iv-110 can be expressed in terms of Eq. IV-30 and Eq. IV-35 thus

$$2\,\frac{f(1-\bar{\lambda})}{2D}\,\rho_f\,\bar{u}_1\,\varepsilon\,e^{st}\,\frac{s-\Omega+\Omega e^{-s\tau_b}}{s} = \frac{2\,\overline{\Delta P_{23}}}{\langle u_g\rangle}\,\delta u_g \qquad \text{IV-114}$$

This term expresses therefore the effect of velocity perturbation in the "light" fluid region on the frictional pressure drop.

Similarly the fourth term in Eq. IV-110 can be expressed as

$$\frac{\Omega}{s-\Omega}\,\frac{f(1-\lambda)}{2D}\,\rho_f u_1\,\varepsilon\,e^{st}\,\frac{s-\Omega+\Omega e^{-s\tau_b}}{s} = \frac{\Omega}{s-\Omega}\,\frac{\overline{\Delta P_{b4}}}{\langle u_g\rangle}\,\delta u_g \qquad \text{IV-115}$$

In view of Eq. IV-79 this term shows the effect of the velocity perturbation in the "light" fluid region on the density perturbation and therefore on the frictional pressure drop in this region.

The physical meaning of the last term in Eq. IV-110 can be explained again by expressing the integral I_5 in Eq. IV-111 by its upper and lower bound (see Appendix C) thus

$$\frac{\Omega}{s-\Omega}\,\frac{f(1-\bar{\lambda})}{2D}\,\frac{G\,\bar{u}_3}{u_1}\,e^{-s(\tau_3-\tau_1)}\,\delta u_1 < I_5 < \frac{\Omega}{s-\Omega}\,\frac{f(1-\bar{\lambda})}{2D}\,G\,e^{-s\tau_b}\,\delta u_1 \qquad \text{IV-116}$$

which in view of Eq. IV-115 and IV-35 can be expressed as:

$$\frac{\Omega}{s-\Omega}\,\frac{\overline{\Delta P_{23}}}{\langle u_g\rangle}\,\frac{\bar{u}_3}{\bar{u}_1}\,e^{-s(\tau_3-\tau_1)}\,\delta u_1 < I_5 < \frac{\Omega}{s-\Omega}\,\frac{\overline{\Delta P_{23}}}{\langle u_g\rangle}\,e^{-s\tau_6}\,\delta u_1 \qquad \text{IV-117}$$

A simple expression for the integral I_5 can be also obtained by using the same approximation that was used in deriving Eq. IV-97 and Eq. IV-107. Thus, if we let $\bar{u}_1/\bar{u}_{g(3)} = \bar{u}_1/\bar{u}_3$ in Eq. IV-111 we obtain after integration

$$I_5^* = \frac{\Omega}{s-\Omega}\,\frac{\overline{\Delta P_{23}}}{\bar{u}_1}\,e^{-s(\tau_3-\tau_1)}\,\delta u_1 \qquad \text{IV-118}$$

The last term in Eq. IV-110 expresses therefore the effect of the inlet velocity perturbation on the density (see Eq. IV-79) and therefore on the frictional pressure drop in the "light" fluid region. Furthermore this effect is delayed by a delay time equal to τ_6 or $(\tau_3-\tau_1)$ depending on whether we use the upper or lower bound for the integral.

By substituting Eq. IV-112, IV-113, IV-114, IV-115 in Eq. IV-110 and expressing the integral I_5 in terms of the approximation given by Eq. IV-118 we obtain for the frictional pressure drop in the "light" fluid region the following expression:

$$\Delta P_{23} = \overline{\Delta P_{23}} - \frac{\overline{\Delta P_{23}}}{(l-\bar{\lambda})} \frac{\bar{u}_1}{\langle u_g \rangle} \delta\lambda + 2 \frac{\overline{\Delta P_{23}}}{\langle u_g \rangle} \delta u_g +$$

$$+ \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_{23}}}{\langle u_g \rangle} \delta u_g - \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_{23}}}{\bar{u}_1} e^{-s(\tau_3-\tau_1)} \delta u_1 \qquad \text{IV-119}$$

where the steady state frictional pressure drop $\overline{\Delta P_{23}}$ is given by Eq. IV-112.

IV.7.5 <u>The Exit Pressure Drop</u>

We can include the effect of the exit pressure drop in the momentum equation. For this purpose we shall define by k_e the coefficient for the exit losses, then the exit pressure drop can be expressed as

$$P_3 - P_4 = \Delta P_{34} = k_e \rho_3 u_3^2 \qquad \text{IV-120}$$

By substituting Eq. IV-69 and Eq. IV-27, both evaluated at $z = l$, and by retaining only the first power in ε we obtain:

$$P_3 - P_4 = k_e G \bar{u}_3 + 2 G \delta u_g + \frac{\Omega}{s-\Omega} G \delta u_g - \qquad \text{IV-121}$$

$$- \frac{\Omega}{s-\Omega} \rho_f \bar{u}_3 e^{-s(\tau_3-\tau_1)} \delta u_1$$

We obtain the steady state exit pressure drop by letting $\varepsilon = 0$ in Eq. IV-121 thus

$$\overline{\Delta P_{34}} = k_e \rho_3 \bar{u}_3^2 = k_e G \bar{u}_3 \qquad \text{IV-122}$$

Consequently Eq. IV-121 can be expressed as

$$\Delta P_{34} = \overline{\Delta P_{34}} + \frac{2 \overline{\Delta P_{34}}}{\bar{u}_3} \delta u_g + \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \delta u_g - \frac{\Omega}{s-\Omega} \frac{\overline{\Delta P_{34}}}{\bar{u}_1} e^{-s(\tau_3-\tau_1)} \delta u_1 \qquad \text{IV-123}$$

The second term in Eq. IV-123 represents the effect of velocity perturbation in the "light" fluid region on the exit pressure drop. The last two terms in Eq. IV-122 can be expressed as

$$\frac{\Omega}{s-\Omega} \overline{\Delta P_{34}} \left\{ \frac{\delta u_g}{\bar{u}_3} - e^{-s(\tau_3-\tau_1)} \frac{\delta u_1}{\bar{u}_1} \right\} = \frac{\overline{\Delta P_{34}}}{\rho_3} \delta \rho_3 \qquad \text{IV-124}$$

where we have taken into account Eq. IV-81. Consequently the last two terms express the effect of the density perturbation on the exit pressure drop.

IV.7.6 The Integrated Momentum Equation

By adding Eq. IV-86, IV-98, IV-108, IV-119 and IV-123 we obtain the integrated momentum equation for the light fluid region thus

$$\begin{aligned}
P_2 - P_4 = {} & \overline{\Delta P_a} + \overline{\Delta P_{bg}} + \overline{\Delta P_{23}} + \overline{\Delta P_{34}} + \\
& + \left\{ (1-\bar{\Lambda}) \langle \rho_g \rangle \right\} \frac{d\,\delta u_g}{dt} + \\
& - \left\{ \frac{\overline{\Delta P_a}}{(1-\bar{\Lambda})} + \frac{\overline{\Delta P_{bg}}}{(1-\bar{\Lambda})} \frac{u_{\ell m}}{u_1} + \frac{\overline{\Delta P_{23}}}{(1-\bar{\Lambda})} \frac{\bar{u}_1}{\langle u_g \rangle} \right\} \delta\Lambda + \\
& + \left\{ \frac{\overline{\Delta P_a}}{u_{\ell m}} + \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{23}}}{\langle u_g \rangle} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \right\} \delta u_g + \\
& + \frac{\Omega}{s-\Omega} \left\{ \frac{\overline{\Delta P_a}}{u_{\ell m}} + \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{23}}}{\langle u_g \rangle} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \right\} \delta u_g - \\
& - \frac{\Omega}{s-\Omega} \left\{ \frac{\overline{\Delta P_a}}{\bar{u}_1} + \frac{\overline{\Delta P_{bg}}}{\bar{u}_1} + \frac{\overline{\Delta P_{23}}}{\bar{u}_1} + \frac{\overline{\Delta P_{34}}}{u_1} \right\} e^{-s(\tau_3-\tau_1)} \delta u_1
\end{aligned} \qquad \text{IV-125}$$

By adding the momentum equations for the "heavy" and "light" fluids we shall obtain the momentum equation for the system whence the characteristic equation for predicting the onset of unstable flow. This will be done in the chapter that follows.

IV.8 Comparison With Previous Results

Before we proceed with the derivations of the characteristic equation, it is of interest to compare the results derived in this chapter with those reported previously in [49, 50, 51, 53, 55]. In this section we shall make comparison with the results of [49, 50 and 51] whereas in the section that follows we shall compare the present results to those of [53, 55].

It was already discussed in Section I.3 that the assumptions made in the present analysis as well as the general formulation of the problem are the same as those reported previously the Wallis and Heasley [50] and Bouré [51] for boiling, two phase system. It was also noted in Section I.3 that the present analysis differs from those reported in [49, 50 and 51] in the following respect: 1) the constitutive equation of state is different and 2) the characteristic equation is different.

The analyses of [49, 50 and 51] were derived for boiling systems, the present investigation is applicable to both subcritical and supercritical pressures. It is emphasized here again that neither this investigation nor those reported in [49, 50 and 51] take into account the effect of relative velocity between the two phases in the boiling region at subcritical pressure.* If the effects of the relative velocity are

*The conditions under which the effects of relative velocity can be neglected are discussed in more detail in [55].

to be taken into account then the momentum and the energy equation, i.e., Eq. IV-2 and IV-3 must be modified. Furthermore, a diffusion equation should be added to the field equations describing the process. An investigation along these lines will be reported separately.

If, in the boiling region, we express the reaction frequency Ω by means of Eq. IV-23, then the density given by Eq. IV-65 becomes identical to that derived first in [49] and to those in [50, 51, 55] using different approaches. We shall examine now Eq. IV-66 which can be expressed also as:

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1}{\bar{u}_g(z)} + \frac{\Omega}{s-\Omega}\,\frac{\rho(z,t)}{\rho_f}\,\frac{\delta u_g}{\bar{u}_g(z)} - \frac{\Omega}{s-\Omega}\,e^{-s(t-\tau_L)}\,e^{-s\tau_b}\,\frac{\delta u_1}{\bar{u}_g(z)} \qquad \text{IV-126}$$

whence, in view of Eq. IV-65, and IV-30, the perturbation can be written as:

$$\delta\rho(z,t) = \frac{\varepsilon e^{st}}{\bar{u}_g(z)}\,\frac{\rho(z,t)}{\rho_f}\,\frac{\Omega}{s-\Omega}\left\{1 - \frac{\Omega(1-e^{-s\tau_b})}{s} - \frac{e^{-(s-\Omega)(t-\tau_L)}\,e^{-s\tau_b}}{s-\Omega}\right\} \qquad \text{IV-127}$$

By adding and subtracting $e^{-s\tau_b}/s$ we can express this relation as

$$\delta\rho(z,t) = \frac{\varepsilon e^{st}}{\bar{u}_g(z)}\,\rho(z,t)\,\Omega\left\{\frac{1-e^{-s\tau_b}}{s} + \frac{e^{-s\tau_b}}{s-\Omega}\left(1-e^{-(s-\Omega)(t-\tau_L)}\right)\right\} \qquad \text{IV-128}$$

If we replace now Ω by Eq. IV-23, then Eq. IV-128 becomes identical to Eq. 38 in the paper by Wallis and Healsey (50) derived using a different approach.

We can insert Eq. IV-65 in Eq. IV-126 and express the latter as

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1}{\bar{u}_g(z)} + \frac{\Omega}{s-\Omega}\,\frac{\rho(z,t)}{\rho_f}\,\frac{\delta u_g}{\bar{u}_g(z)} - \frac{\Omega}{s-\Omega}\left[\frac{\rho(z,t)}{\rho_f}\right]^{s/\Omega} e^{-s\tau_b}\,\frac{\delta u_1}{\bar{u}_g(z)} \qquad \text{IV-129}$$

if the density terms which appear on the right hand side of Eq. IV-129 are approximated by the steady state relation, i.e., by

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1}{\bar{u}_g(z)} \qquad \text{IV-130}$$

as was done in (51) we obtain

$$\frac{\rho(z,t)}{\rho_f} = \frac{\bar{u}_1}{\bar{u}_g(z)} + \frac{\varepsilon e^{st}}{\bar{u}_1}\,\Omega\left\{\frac{s-\Omega+\Omega e^{-s\tau_b}}{s(s-\Omega)}\left(\frac{\bar{u}_1}{\bar{u}_g(z)}\right)^2 - \frac{e^{-s\tau_b}}{s-\Omega}\left(\frac{\bar{u}_1}{\bar{u}_g(z)}\right)^{\frac{s}{\Omega}+1}\right\} \qquad \text{IV-131}$$

which is equivalent to Eq. 5, Appendix A of Boure's report (51).

Apart from the difference in the equations of state used in this analysis, the difference between the present results and those of (50, 51) is in the handling the momentum equation. In (50) the momentum equation was not integrated along the duct, it was first integrated by Bouré (51). Indeed, it can be shown, that after some rearrangement, Eq. IV-88, IV-100 and IV-110 can be put in the form of those given in (51). In (51) the integration of the momentum equation lead to a characteristic equation in the form of an exponential polynomial of the fourth (or higher) order.

In the present analysis we have introduced various definitions for the mean, for the average and for the log mean density as well as for the velocities in the "light" fluid region which enabled us to give physical interpretation to the various terms in the integrated momentum equation. It will be seen in what follows that these relations, together with the approximation used in deriving Eq. IV-97, IV-107 and IV-118, result in a characteristic equation given by an exponential polynomial of the third order. It will be seen also in what follows that these results will enable us to derive stability criteria and stability maps which, previously, were not available in the literature.

IV.9 The Density Propagation Equation

It is of interest to note an alternate way for determining the density perturbation.

If we substitute Eq. IV-21 in Eq. IV-19 we obtain:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial z} = -\rho \Omega \qquad \text{IV-132}$$

This equation was called the energy equation in (51) where it was first derived. Several remarks are relevant here.

We note first that Eq. IV-132 predicts the propagation of the density caused by the source term Ω. A "void propagation equation" was formulated in (53 and 55) in terms of kinematic waves which predicts the propagation of density perturbations through a two-phase system.

This void propagation equation takes into account the effect of the relative velocity between the two phase as well as the effect of the non-uniform velocity and concentration profiles in the two phase mixture. It can be easily shown that if these effects are neglected the void propagation equation can be reduced to Eq. IV-132.

We note also that Eq. IV-132 is of the same form as the continuity for a given species in a multicomponent, chemical reaction system. In chemical kinetics the source term in Eq. IV-132 is referred to as the reaction frequency. It is for this reason that in [53, 55] the term was called the "characteristic frequency."

Finally, we note that Eq. IV-132 is a first order partial differential equation which can be solved by the standard method used in Sections III-3 and IV-4. Indeed following this procedure, used already in [53 and 55], one can derive Eq. IV-66 and Eq. IV-68.

V. The Characteristic Equation

V.1. The Momentum Equation for the System

The momentum for the "heavy" fluid is given by Eq. III-33, whereas that for the "light" fluid is given by Eq. IV-125. By adding these two equations, we obtain the momentum equation for the system.

We note that if the downstream pressure P_4, is constant we can express the overall pressure drop, i.e., the external pressure drop of the system as a steady state term and a pressure perturbation caused by the inlet flow. Thus

$$P_0 - P_4 = \overline{\Delta P}_{ex} + \frac{\partial \overline{\Delta P}_{ex}}{\partial u_1} \delta u_1 \qquad \text{V-1}$$

where the second term on the right hand side is determined by the pump characteristics and has a negative value.

By adding Eq. III-33, Eq. IV-125 and Eq. V-1 we obtain the integrated momentum equation for the heated duct, thus

$$\begin{aligned}
\overline{\Delta P}_{ex} = {} & \overline{\Delta P}_{01} + \overline{\Delta P}_{12} + \overline{\Delta P}_{b4} + \overline{\Delta P}_{a} + \overline{\Delta P}_{bg} + \overline{\Delta P}_{23} + \overline{\Delta P}_{34} + \\
& + \left\{ \rho_f \bar{\lambda} \right\} \frac{d\,\delta u_1}{dt} + \left\{ \langle \rho_g \rangle (l - \bar{\lambda}) \right\} \frac{d\,\delta u_g}{dt} + \\
& + \left\{ \frac{\partial \overline{\Delta P}_{01}}{\partial u_1} + \frac{\partial \Delta P_{12}}{\partial u_1} - \frac{\partial \overline{\Delta P}_{ex}}{\partial u_1} \right\} \delta u_1 + \\
& + \left\{ \frac{\overline{\Delta P}_{a}}{u_{gm}} + \frac{2\overline{\Delta P}_{23}}{\langle u_g \rangle} + \frac{2\overline{\Delta P}_{34}}{\bar{u}_3} \right\} \delta u_g +
\end{aligned} \qquad \text{V-2}$$

$$+\frac{\Omega}{s-\Omega}\left\{\frac{\overline{\Delta P_a}}{u_{\ell m}}+\frac{\overline{\Delta P_{bg}}}{u_m}+\frac{\overline{\Delta P_{23}}}{\langle u_g\rangle}+\frac{\overline{\Delta P_{34}}}{\bar{u}_3}\right\}\delta u_g -$$

$$-\frac{\Omega}{s-\Omega}\left\{\frac{\overline{\Delta P_a}}{\bar{u}_1}+\frac{\overline{\Delta P_{bg}}}{\bar{u}_1}+\frac{\overline{\Delta P_{23}}}{\bar{u}_1}+\frac{\overline{\Delta P_{34}}}{\bar{u}_3}\right\}e^{-s(\tau_3-\tau_1)}\delta u_1 -$$

$$-\frac{\overline{\Delta P_a}}{\ell-\lambda}\delta\Lambda = 0$$

We obtain the steady state pressure drop for the system by letting the perturbations go to zero, thus

$$\overline{\Delta P_{ex}} = \overline{\Delta P_{01}} + \overline{\Delta P_{12}} + \overline{\Delta P_{bf}} + \overline{\Delta P_a} + \overline{\Delta P_{bg}} + \overline{\Delta P_{23}} + \overline{\Delta P_{34}} \qquad \text{V-3}$$

By subtracting Eq. V-3 from Eq. V-2 we obtain the perturbed form of the momentum equation, thus

$$\left\{\rho_f \bar{\lambda}\right\} \frac{d\,\delta u_1}{dt} + \left\{\langle\rho_g\rangle(1-\bar{\lambda})\right\} \frac{d\,\delta u_g}{dt} +$$

$$+ \left\{ \frac{\partial \overline{\Delta P_{01}}}{\partial u_1} + \frac{\partial \overline{\Delta P_{12}}}{\partial u_1} + \left| \frac{\partial \overline{\Delta P_{ex}}}{\partial u_1} \right| \right\} \delta u_1 +$$

$$+ \left\{ \frac{\overline{\Delta P_a}}{u_{fm}} + \frac{2\,\overline{\Delta P_{23}}}{\langle u_g\rangle} + \frac{2\,\overline{\Delta P_{34}}}{\bar{u}_3} \right\} \delta u_g + \qquad \text{V-4}$$

$$+ \frac{\Omega}{s-\Omega} \left\{ \frac{\overline{\Delta P_a}}{u_{fm}} + \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{23}}}{\langle u_g\rangle} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \right\} \delta u_g -$$

$$- \frac{\Omega}{s-\Omega} \left\{ \frac{\overline{\Delta P_a}}{\bar{u}_1} + \frac{\overline{\Delta P_{bg}}}{\bar{u}_1} + \frac{\overline{\Delta P_{23}}}{\bar{u}_1} + \frac{\overline{\Delta P_{34}}}{\bar{u}_1} \right\} e^{-s(\tau_3-\tau_1)}\, \delta u_1$$

$$- \frac{\overline{\Delta P_a}}{\Omega(1-\bar{\lambda})}\, \Omega\, \delta\lambda = 0$$

The formulation is now essentially complete because Eq. V-4 is the expression which gives the response of the system to the initial flow pertubation as function of the influence coefficients defined below.

The influence coefficients F_1 and F_2 represent the mass of the "heavy" fluid and of the "light" fluid respectively, thus

$$F_1 = \left\{\rho_f \bar{\lambda}\right\} = M_f \qquad \text{V-5}$$

and

$$F_2 = \left\{\langle\rho_g\rangle(1-\bar{\lambda})\right\} = M_g \qquad \text{V-6}$$

The coefficient F_3 describes the effect of the inlet flow variation on the pressure drops in the "heavy" fluid region, thus

$$F_3 = \frac{\partial \overline{\Delta P_{01}}}{\partial u_1} + \frac{\partial \overline{\Delta P_{12}}}{\partial u_1} + \left| \frac{\partial \Delta P_{ex}}{\partial u_1} \right| = \frac{2 \Delta P_{01}}{\bar{u}_1} + \frac{2 \Delta P_{12}}{\bar{u}_1} + \left| \frac{\partial \Delta P_{ex}}{\partial u_1} \right| \qquad \text{V-7}$$

This coefficient, which is well known from studies of the transient response of single phase flow systems, has always a positive value.

The coefficient F_4 shows the effect of the velocity perturbation in the "light" fluid region on the pressure drops in that region, thus

$$F_4 = \frac{\overline{\Delta P_a}}{u_{em}} + \frac{2 \overline{\Delta P_{23}}}{\langle u_g \rangle} + \frac{2 \overline{\Delta P_{34}}}{\bar{u}_3} \qquad \text{V-8}$$

It is of considerable importance to note that each pressure drop is differentiated and is weighed therefore by a <u>different</u> velocity. This important result is a consequence of the integration of the momentum equation, i.e., of the <u>distributed parameter</u> analysis. We note that in the <u>"lumped" parameter</u> analysis the three pressure drops in Eq. V-8 would have been divided by the same velocity, say by the velocity $\bar{u}_3$ at the exit from the test section as is most often the case for analyses reported in the literature.

The influence coefficients F_5 and F_6 are given by

$$F_5 = \frac{\overline{\Delta P_a}}{u_{\ell m}} + \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{23}}}{\langle u_g \rangle} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \qquad \text{V-9}$$

and

$$F_6 = \frac{\overline{\Delta P_a}}{\bar{u}_1} + \frac{\overline{\Delta P_{bg}}}{\bar{u}_1} + \frac{\overline{\Delta P_{23}}}{\bar{u}_1} + \frac{\overline{\Delta P_{34}}}{\bar{u}_1} \qquad \text{V-10}$$

It can be seen from Eq. IV-69 and Eq. V-2 that these two coefficients account for the effect of the density perturbation on the various pressure drops in the "light" fluid region. Note, that the density perturbation depends on both $\bar{u}_g(z)$ and $\bar{u}_1$. Two observations are noteworthy. First, the coefficient F_5 shows that the effects of the velocity perturbation on the "light" fluid region are weighed by various velocities. This, again, is a consequence of the distributed parameter approach. Two, the exponential which multiplies the coefficient F_6 indicates that the effects of the inlet perturbation are delayed by the delay time $\tau_3 - \tau_1$.

Finally the coefficient F_7, defined by

$$F_7 = \frac{1}{\Omega} \frac{\partial \Delta P_a}{\partial (1-\bar{\lambda})} = \frac{\overline{\Delta P_a}}{\Omega (1-\bar{\lambda})} \qquad \text{V-11}$$

shows the effect of the space lag perturbation on the acceleration pressure drop in the "light" fluid region. It is important to notice here that in Eq. III-33 and Eq. IV-125 all other terms which are differentiated with respect to the length cancel each other in the momentum equation for the system. This result could not have been anticipated in a "lumped" parameter analysis. Indeed, in several studies of boiling systems using the "lumped" parameter approach these terms were introduced and retained in the analysis. In view of the foregoing, the results and conclusions based on such formulations can be considered as spurious.

By introducing Eq. V-5 through V-11 in Eq. V-4 the perturbed momentum equation for the heated duct can be expressed by

$$F_1 \frac{d\delta u_1}{dt} + F_2 \frac{d\delta u_g}{dt} + F_3\, \delta u_1 + F_4\, \delta u_g + \frac{\Omega}{s-\Omega} F_5\, \delta u_g - \frac{\Omega}{s-\Omega} F_6\, e^{-s(\tau_3-\tau_1)} \delta u_1 - F_7\, \Omega\, \delta\lambda = 0 \qquad \text{V-12}$$

Before deriving the characteristic equation it will be instructive to express the perturbations in Eq. V-12 in terms of the perturbations of the inlet flow and of the space lag. Taking into account Eq. IV-30 we can express Eq. V-12 as

$$\left\{F_1 + F_2\right\} \frac{d\delta u_1}{dt} + \left\{F_3 + F_4\right\} \delta u_1 + \frac{\Omega}{s-\Omega}\left\{F_5 - F_6\, e^{-s(\tau_3-\tau_1)}\right\} \delta u_1 - F_2\, \Omega \frac{d\delta\lambda}{dt} - \left\{F_4 + \frac{\Omega}{s-\Omega} F_5 + F_7\right\} \Omega\, \delta\lambda = 0 \qquad \text{V-13}$$

It can be clearly seen from Eq. V-13 that the dynamic response of the heated channel depends upon both the inlet flow perturbation and the variation of the space lag. This latter effect is an example of a fluctuation which occurs <u>inside</u> the system. It was discussed in Sections II-4 and II-7 that such fluctuation have a destabilizing effect. The destabilizing effect which the space lag variation has in combustion systems and in boiling systems has been already demonstrated in [48, 52] and [50, 51] among others. Equation V-4 shows that, at supercritical pressures, the space lag variation has a similarly destabilizing effect. Furthermore, the negative sign in the third term on the left hand side of Eq. V-13 shows the destabilizing effect of the inlet velocity perturbation. We have noted already that this effect stems from the density perturbation in the "light" fluid region.

V.2 The Characteristic Equation

In view of the definitions of the inlet velocity perturbation and of the space lag perturbation given by Eq. III-7 and Eq. IV-30 respectively, we can express Eq. V-13 as

$$\varepsilon e^{st}\Big\{ s[F_1+F_2] + F_3 + F_4 + \frac{\Omega}{s-\Omega} F_5 - \frac{\Omega}{s-\Omega} F_6 e^{-s(\tau_5-\tau_1)} - $$

$$- s F_2 \Omega\left(\frac{1-e^{-s\tau_6}}{s}\right) - \left[F_4 + \frac{\Omega}{s-\Omega} F_5 + F_7\right]\Omega\left(\frac{1-e^{-s\tau_6}}{s}\right)\Big\} = 0 \qquad \text{V-14}$$

From this relation we obtain the characteristic equation by noting that since $\varepsilon \neq 0$ the sum of the terms within the bracket must be equal to zero. Thus, after multiplying by $(s-\Omega)$ and after some rearrangement we obtain the characteristic equation for the heated duct:

$$s^2[F_1+F_2] + s[F_3+F_4-\Omega(F_1+F_2)] - \Omega[F_3+F_4-F_5] - \Omega F_6 e^{-s(\tau_3-\tau_1)} -$$

V-15

$$-\Omega\left(\frac{1-e^{-s\tau_b}}{s}\right)\left\{s^2F_2 + s[F_4+F_7-\Omega F_2] - \Omega[F_4+F_7-F_5]\right\} = 0$$

It can be seen that the characteristic equation is a third order polynomial with two time delays. From the definitions of the influence coefficients we have the following relations for the various terms which appear in Eq. V-15.

$$F_1+F_2 = M_f + M_g = \rho_f\bar{\lambda} + \langle\rho_g\rangle(1-\bar{\lambda}) \qquad \text{V-16}$$

$$F_3+F_4 = \frac{2\overline{\Delta P_{01}}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\frac{\partial\overline{\Delta P_{ex}}}{\partial\bar{u}_1}\right| + \frac{\overline{\Delta P_a}}{u_{lm}} + \frac{2\overline{\Delta P_{23}}}{\langle u_g\rangle} + \frac{2\overline{\Delta P_{34}}}{\bar{u}_3} \qquad \text{V-17}$$

$$F_5 = \frac{\overline{\Delta P_a}}{u_{lm}} + \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{23}}}{\langle u_g\rangle} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \qquad \text{V-18}$$

$$F_3+F_4-F_5 = \frac{2\overline{\Delta P_{01}}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\frac{\partial\overline{\Delta P_{ex}}}{\partial u_1}\right| + \frac{\overline{\Delta P_{23}}}{\langle u_g\rangle} - \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \qquad \text{V-19}$$

V-20

$$F_4+F_7 = \frac{\overline{\Delta P_a}}{u_{lm}} + \frac{2\overline{\Delta P_{23}}}{\langle u_g\rangle} + \frac{2\overline{\Delta P_{34}}}{\bar{u}_3} + \frac{\overline{\Delta P_a}}{\Omega(1-\bar{\lambda})}$$

V-21

$$F_4+F_7-F_5 = \frac{\overline{\Delta P_a}}{\Omega(1-\bar{\lambda})} + \frac{\overline{\Delta P_{23}}}{\langle u_g\rangle} - \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3}$$

V-22

$$F_6 = \frac{\overline{\Delta P_a}}{\bar{u}_1} + \frac{\overline{\Delta P_{bg}}}{\bar{u}_1} + \frac{\overline{\Delta P_{23}}}{\bar{u}_1} + \frac{\overline{\Delta P_{34}}}{u_1}$$

It was discussed in Section II-6 that the characteristic equation predicts the value of S as function of the pressure terms given by Eq. V-16 through V-22. In general s is a complex number $S = a + i\omega$, the real part gives the amplification coefficient of the particular oscillation mode, whereas the imaginary part represents the angular frequency ω. Since the original perturbation of the inlet velocity was assumed to be of the form $\delta u_1 = \varepsilon e^{st}$, a given oscillating mode will be stable, metastable or unstable depending on whether the real part of S is less, equal or larger than zero, i.e., whether $a < 0$, $a = 0$ or $a > 0$.

A general study of the flow behaviour entails an investigation of conditions leading to aperiodic as well as to periodic phenomena. The first pertains to the possibility of flow excursion whereas the second pertains to the onset of flow oscillations. Following the standard procedure we shall study aperiodic phenomena by considering the case of S = a with $\omega = 0$. Again, following the standard procedure we shall study periodic phenomena by setting $S = i\omega$ ($a = 0$, $\omega \neq 0$) in the characteristic equation. Such an approach will enable us to determine the stability boundary which defines regions of stable and of oscillating behaviour in a stability map. In the study of the oscillatory phenomena we shall consider separately the case of high subcooling and the case of low subcooling. The stability problem at intermediate subcoolings will be considered in a separate report.

VI. Excursive Instability

VI.1 Derivation of the Stability Criterion

The study of excursive, i.e., of aperiodic instabilities is conducted by considering the exponent s of the velocity perturbation to be real, i.e., by letting the angular frequency ω of the disturbance be zero. It follows then from Eq. V-15 that for small values of s, we have the following relation:

$$sF_1 + sF_2(1-\Omega\tau_6) + F_3 + F_4(1-\Omega\tau_6) -$$

$$-\left(1+\frac{s}{\Omega}\right)F_5(1-\Omega\tau_6) + \left(1+\frac{s}{\Omega}\right)F_6(1-s\tau_6) - F_7\Omega\tau_6 = 0 \qquad \text{V1-1}$$

whence after rearrangement:

$$s\left\{F_1 + F_2(1-\Omega\tau_6) - \frac{F_5}{\Omega}(1-\Omega\tau_6) + \frac{F_6}{\Omega}(1-\Omega\tau_6)\right\} + \qquad \text{V1-2}$$

$$+\left\{F_3 + F_4(1-\Omega\tau_6) - F_5(1-\Omega\tau_6) + F_6 - F_7\Omega\tau_6\right\} = 0$$

which is of the form

$$sA^* + B^* = 0 \qquad \text{V1-3}$$

Equation VI-2 predicts the value of the exponent S in terms of the influence coefficients. Since the inlet velocity perturbation is of the form of $\delta u = \varepsilon e^{st}$, and since the coefficient A* is positive and exponent S is real, Equation VI-3 indicates that the flow will be stable, i.e., the disturbance will decrease with time if B* is positive, thus from Equation VI-2

$$B^* = F_3 + F_4(1-\Omega\tau_b) - F_5(1-\Omega\tau_b) + F_6 - F_7\,\Omega\tau_b > 0 \qquad \text{VI-4}$$

If B is negative then Equation VI-3 indicates that s will be real and positive, consequently any flow disturbance will be amplified with time resulting in flow excursions. Substituting the definitions for the influence coefficients given by Equation V-5 through Equation V-11 we can express Equation VI-4 in terms of steady state pressure drops, thus

$$\frac{2\overline{\Delta P}_{01}}{\bar{u}_1} + \frac{2\overline{\Delta P}_{12}}{\bar{u}_1} + \left|\frac{\partial \Delta P_{ex}}{\partial u_1}\right| + \left\{\frac{\overline{\Delta P}_{23}}{\langle u_g \rangle} - \frac{\overline{\Delta P}_{bg}}{u_m} + \frac{\overline{\Delta P}_{34}}{\bar{u}_3}\right\}(1-\Omega\tau_b) + \qquad \text{VI-5}$$

$$+ \frac{\overline{\Delta P}_a}{\bar{u}_1} + \frac{\overline{\Delta P}_{bg}}{\bar{u}_1} + \frac{\overline{\Delta P}_{23}}{\bar{u}_1} + \frac{\overline{\Delta P}_{34}}{\bar{u}_1} - \frac{\overline{\Delta P}_a}{\Omega(\ell-\bar{\lambda})}\,\Omega\tau_b \geq 0$$

This inequality can be cast in a compact form by means of the identities listed below:

$$\frac{d\overline{\Delta P_{01}}}{du_1} = \frac{2\overline{\Delta P_{01}}}{\bar{u}_1} \qquad \text{V1-6}$$

$$\frac{d\overline{\Delta P_{12}}}{du_1} = \frac{3\overline{\Delta P_{12}}}{\bar{u}_1} \qquad \text{V1-7}$$

$$\frac{d\overline{\Delta P_{bf}}}{du_1} = \frac{\overline{\Delta P_{bf}}}{\bar{u}_1} \qquad \text{V1-8}$$

$$\frac{d\overline{\Delta P_{a}}}{du_1} = \frac{\overline{\Delta P_{a}}}{u_{1m}} - \frac{\overline{\Delta P_{a}}}{\Omega(1-\bar{\lambda})}\Omega\tau_b \qquad \text{V1-9}$$

$$\frac{d\overline{\Delta P_{bg}}}{du_1} = \frac{\overline{\Delta P_{bg}}}{\bar{u}_1} - \frac{\overline{\Delta P_{bg}}}{u_m}(1-\Omega\tau_b) - \frac{\overline{\Delta P_{bf}}}{\bar{u}_1} \qquad \text{V1-10}$$

$$\frac{d\overline{\Delta P_{23}}}{du_1} = \frac{\overline{\Delta P_{23}}}{\bar{u}_1} + \frac{\overline{\Delta P_{23}}}{\langle u_g\rangle}(1-\Omega\tau_b) - \frac{\overline{\Delta P_{12}}}{\bar{u}_1} \qquad \text{V1-11}$$

$$\frac{d\overline{\Delta P_{34}}}{du_1} = \frac{\overline{\Delta P_{34}}}{\bar{u}_1} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3}(1-\Omega\tau_b) \qquad \text{V1-12}$$

These relations can be easily derived from the definitions of the steady state pressure drops.

Substituting Eq. VI-6 through V1-12 in Eq. V1-5 we obtain the stability criterion:

$$\frac{d}{du_1}\left\{\overline{\Delta P_{01}}+\overline{\Delta P_{12}}+\overline{\Delta P_{bf}}+\overline{\Delta P_{a}}+\overline{\Delta P_{bg}}+\overline{\Delta P_{23}}+\overline{\Delta P_{34}}\right\}+\left|\frac{d\overline{\Delta P_{ex}}}{du_1}\right|\geq 0 \qquad \text{V1-13}$$

which can be expressed also in terms of the total mass flow rate W, thus

$$\frac{d\overline{\Sigma\Delta P}}{dW}+\left|\frac{d\overline{\Delta P_{ex}}}{dW}\right|\geq 0 \qquad \text{V1-14}$$

For boiling systems, this simple criterion was first derived by Ledinegg [24] using a different approach, it was analyzed further in [25 through 47] and [51]. The results of this analysis show that this "Ledinegg instability" can occur also at supercritical pressures. The significance of the stability criterion given by Eq. V1-14, can be best analyzed by considering the steady state ΔP-W relation for the heated duct. This will be done in the section that follows.

V1.2 Significance of the Stability Criterion

If, for simplicity, we neglect the effect of the gravitational force and if we express the steady state pressure drops in the heated duct in terms of the total mass flow W, and of the total heat input $\dot{Q}$, we have the following relations:

$$\overline{\Delta P_{01}} = k_i \rho_f \bar{u}_1^2 = \frac{k_i}{A_c^2}\frac{W^2}{\rho_f} \qquad \text{V1-15}$$

$$\overline{\Delta P_{12}} = \frac{f\bar{\lambda}}{2D}\,\rho_f\,\bar{u}_1^2 \doteq \frac{f\ell}{2DA^2}\,\frac{W^3\Delta i_{21}}{\rho_f\,\dot{Q}}$$

V1-16

$$\overline{\Delta P_a} = G(\bar{u}_3 - \bar{u}_1) = G\Omega(\ell-\bar{\lambda}) = G\left(\frac{dv}{di}\right)_P \frac{\dot{Q}}{A_c\ell}(\ell-\bar{\lambda}) =$$

V1-17

$$= \frac{W}{A_c}\left(\frac{dv}{di}\right)_P \frac{\dot{Q}}{A_c} - \frac{W^2}{A_c^2}\left(\frac{dv}{di}\right)_P \Delta i_{21}$$

$$\overline{\Delta P_{23}} = \frac{f(\ell-\bar{\lambda})}{2D}\,G\langle u_g\rangle = \frac{f(\ell-\bar{\lambda})}{2D}\,G\left[\bar{u}_1 + \frac{\Omega(\ell-\bar{\lambda})}{2}\right] =$$ V1-18

$$= \frac{f\ell}{2D}\left[1 - \frac{W\Delta i_{21}}{\dot{Q}}\right]\frac{W}{A_c}\left[\frac{W}{A_c\rho_f} + \frac{1}{2}\left(\frac{dv}{di}\right)_P \frac{\dot{Q}}{A_c}\left(1 - \frac{W\Delta i_{21}}{\dot{Q}}\right)\right]$$

$$\overline{\Delta P_{34}} = k_e\,\rho_3\,\bar{u}_3^2 = k_e\,G\,\bar{u}_3 = k_e\frac{W}{A_c}\left[\bar{u}_1 + \Omega(\ell-\bar{\lambda})\right] =$$ V1-19

$$= k_e\frac{W}{A_c}\left[\frac{W}{A_c\rho_f} + \left(\frac{dv}{di}\right)_P \frac{\dot{Q}}{A_c}\left(1 - \frac{W\Delta i_{21}}{\dot{Q}}\right)\right]$$

The total pressure drop for the heated duct is obtained by adding Eq. V1-15 through V1-19, thus

$$\sum \Delta P = a^*\frac{W^3}{\dot{Q}} - b^* W^2 + c^*\dot{Q}W$$ V1-20

where the coefficients a, b and c are given by

$$a^* = \frac{1}{2}\,\frac{f\ell}{2DA^2}\left(\frac{dv}{di}\right)_P \Delta i_{21}^2$$ V1-21

$$b^* = \frac{f\ell}{2DA^2}\left\{\left(\frac{dv}{di}\right)_p \Delta i_{21}\left(1+\frac{2D}{f\ell}\right) - \frac{1}{\rho_f}\left(1+\frac{k_i 2D}{f\ell}\right) + \right. \tag{V1-22}$$

$$\left. + \frac{k_e 2D}{f\ell}\left[\left(\frac{dv}{di}\right)_p \Delta i_{21} - \frac{1}{\rho_f}\right]\right\}$$

$$c^* = \frac{f\ell}{2DA^2}\left\{\frac{2D}{f\ell} + \frac{1}{2} + k_e \frac{2D}{f\ell}\right\}\left(\frac{dv}{di}\right)_p \tag{V1-23}$$

It should be noted that Eq. V1-20 is applicable to subcritical as well as to supercritical pressures. By assigning the proper expression to (dv/di), which we obtain from the equation of state, we can differentiate the process of boiling at subcritical pressures from the process of heat transfer at supercritical pressures. Thus, for boiling at subcritical pressures we have from Eq. IV-15

$$\left(\frac{dv}{di}\right)_p = \frac{\Delta v_{fg}}{\Delta i_{fg}} \tag{V1-24}$$

whereas at supercritical pressures we obtain from Eq. IV-8

$$\left(\frac{dv}{di}\right)_p = \frac{R}{P c_p} \tag{V1-25}$$

When Eq. V1-24 is substituted in Eq. V1-21, V1-22 and V1-23, then Eq. V1-20 becomes the pressure drop relation first derived and discussed by Schnackenberg (25) and Ledinegg (24) for boiling systems. For supercritical pressures Eq. V1-20 was derived by the writer(63) (see also Appendix B).

It can be seen from Eq. Vl-20 that whether in boiling at subcritical pressures or in heating at supercritical pressures the steady state pressure drop in the heated duct has the <u>same</u> cubic dependence upon the total mass flow rate. This important conclusion from analysis is indeed supported by the experimental data reported by Krasiakova and Glusker [18] for water in forced flow through a circular heated duct. Figure Vl-1, which is reproduced from [18], shows that in boiling at subcritical pressures (P = 140 bars) as well as in heating at supercritical pressures (P = 226 bars) the pressure drop in the heated duct has the same cubic dependence upon the mass flow rate. It could be anticipated therefore that the system will have similar dynamic characteristics at these two pressure levels. This is indeed the case as it will be shown later.

The significance of the stability criterion given by Eq. Vl-14 can be best analyzed by plotting Eq. Vl-20 together with the pump characteristic on the same graph. Figure Vl-2 shows such a plot together with three possible flow delivery characteristics, i.e., 1) constant pressure drop delivery system, 2) constant flow rate delivery system and 3) delivery system specified by the pump characteristics. The intersection of the pressure drop for the heated duct with the pressure drop curve of the delivery system determines the operating point of the system. The stability criterion given by Eq. Vl-14 indicates that for some of these operating points the system may be unstable with respect to some small flow perturbations. In order to show this we shall consider each flow delivery system separately.

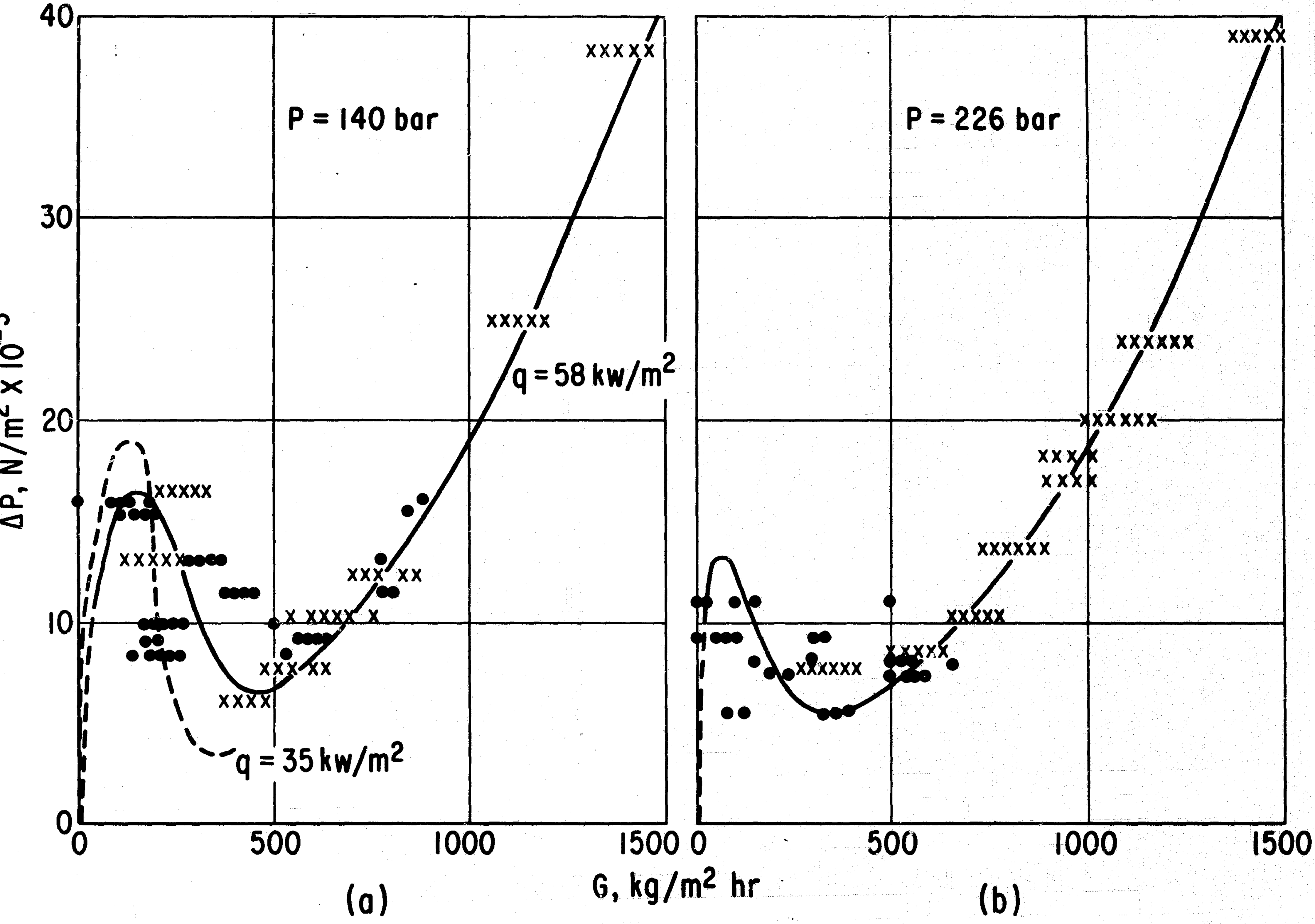

FIGURE VI-1 HYDRAULIC CHARACTERISTICS OF WATER FLOWING THROUGH A HEATED CIRCULAR DUCT AT SUBCRITICAL AND AT SUPERCRITICAL PRESSURES (DATA OF KRASIAKOVA AND OF GLUSKER)

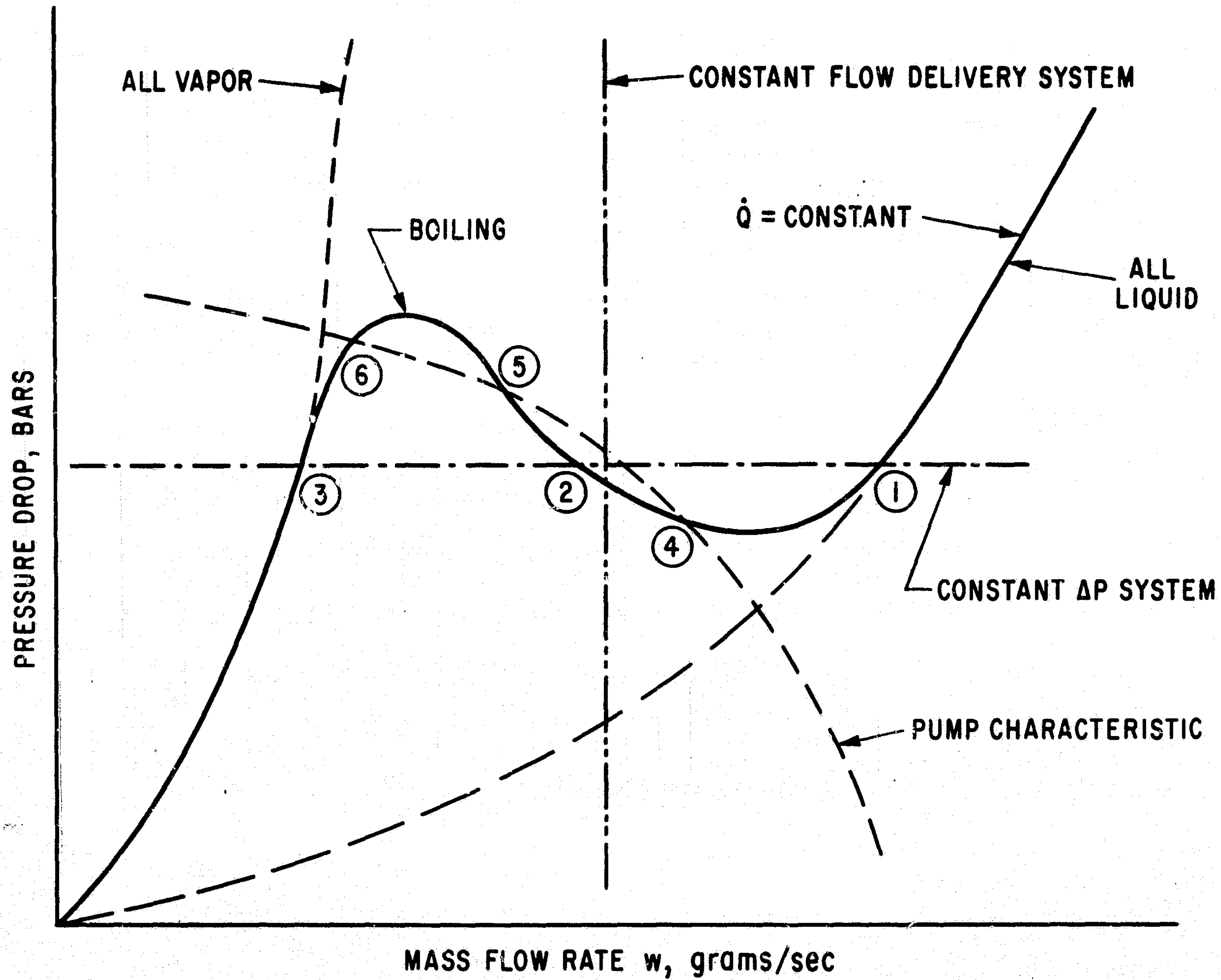

FIGURE VI-2 EXCURSIVE FLOW INSTABILITY

V1.2.1 Constant Pressure Drop Supply System

The operating point for a constant pressure drop delivery system are indicated by points 1, 2 and 3 in Figure V1-2. The stability criterion given by Eq. V1-14 indicates that operation at points 1 and 3 will be stable whereas that at point 2 will be unstable. For example, if a points 1 and 3 the flow is slightly increased the pressure drop of the heated duct increases, i.e., the "demand" curve of the system increases above the "supply" curve of the delivery, consequently the flow will return to its original value. Similarly, if at points 1 and 3 the flow is decreased the pressure drop of the delivery will be above that required by the heated duct resulting in an increased flow and return to the original operating point. However, the operation at point 2 will be unstable with respect to either a flow increase or a flow decrease. If the flow is slightly increased at point 2 the external system supplies more pressure drop than that required to maintain the flow. Consequently the flow rate will increase until the new operating point is reached. Similarly, if the flow is decreased at point 2 more pressure drop is required to maintain the flow than is being supplied by the delivery system. Consequently the flow will decrease until the new operating point 3 is reached.

The preceeding considerations can be expressed in a mathematical form by noting that for a constant pressure drop delivery system Eq. V1-20 reduces to

$$\frac{d \sum \overline{\Delta P}}{dW} > 0 \qquad \text{V1-26}$$

which in view of Eq. V1-20 becomes

$$\frac{d\,\Sigma\overline{\Delta P}}{dw} = 3a^{*}\frac{w^{2}}{\dot{Q}} - 2b^{*}w + c^{*}\dot{Q} > 0 \qquad \text{V1-27}$$

It can be seen from Eq. V1-26 that flow stability requires an increasing pressure drop with flow rate. This is indeed the characteristic of most flow systems. However, the negative term in Eq. V1-27 indicates that for boiling systems as well as for systems at supercritical pressures the pressure drop may decrease with flow rate resulting in flow excursion.

Instead of the stability criterion given by Eq. V1-26 one can introduce the <u>coefficient of stability</u> S, apparently first proposed by Schnackenberg [25] and defined by

$$S = \frac{w}{\Delta P}\left(\frac{d\,\Sigma\overline{\Delta P}}{dw}\right)_{\dot{Q}} \qquad \text{V1-28}$$

which in view of Eq. V1-20 and V1-27 can be expressed as

$$S = \frac{3 - 2\frac{b^{*}}{a^{*}}\frac{\dot{Q}}{w} + \frac{c^{*}}{a^{*}}\left(\frac{\dot{Q}}{w}\right)^{2}}{1 - \frac{b^{*}}{a^{*}}\frac{\dot{Q}}{w} + \frac{c^{*}}{a^{*}}\left(\frac{\dot{Q}}{w}\right)^{2}} \qquad \text{V1-29}$$

where the coefficients a, b and c are given by Eq. V1-21, V1-22 and V1-23.

As observed by Schneckenberg [25] the stability coefficient S, defined by Eq. V1-28, represents the per cent change in the pressure drop by a 1%

variation in the mass flow rate. It can be seen from Eq. VI-28 and VI-26 that for stable flow S must be positive, thus

$$S > 0 \qquad \text{VI-30}$$

V1.2.2 Constant Flow Delivery System

The operating point for a constant flow delivery system is given by the intersection of the pressure supply with pressure demand curves. It can be seen from Figure VI-2 that for such a system

$$\left| \frac{d \overline{\Delta P_{ex}}}{dW} \right| = \infty \qquad \text{VI-31}$$

whence Eq. VI-14 indicates that for such a system no flow excursions are possible.

V1.2.3 Delivery Specified by Pump Characteristics

The operating points for a system whose flow delivery is specified by the characteristics of the pump are shown as points 4, 5 and 6 on Figure VI-2. Using exactly the same arguments as those used in discussing a constant pressure drop delivery system, it can be shown that the operating points 4 and 6 are stable whereas operating point 5 is unstable with respect to small flow disturbances. At this latter point any flow perturbation will cause a flow excursion to either point 4 or to point 6.

VI.3 The Effects of Various Parameters and the Methods for Improving Flow Stability

The effects which various parameters have on the propensity for flow excursions can be evaluated by examing Eq. V1-14, V1-21, V1-22, V1-23 and Eq. V1-27. It can be seen that the variation of any parameter which tends to increase the value of the coefficient b given by Eq. V1-22 will have a destabilizing effect. Consequently, increasing the value of the exit pressure drop coefficient k_e is destabilizing whereas the flow can be stabilized by a high inlet pressure drop, i.e., by appropriate orificing. In view of Eq. V1-24 and V1-25 it can be also seen that increasing the system pressure will have stabilizing effect whereas a decrease in system pressure has the opposite effect. Furthermore, the flow can be also stabilized by changing the pump characteristics.

Before closing the discussion of excursive instabilities it will be instructive to illustrate the destabilizing effect of the compressibility of the fluid in the heated duct. It was discussed in Section 1.3 that the instability mechanism which is analyzed in this paper is based on the effects of time lag and of density variations in the heated duct.

For simplicity we shall consider only the effect of the frictional pressure drop in a system with zero inlet subcooling, i.e., with $\Delta i_{21} = 0$. For such a system Eq. III-20 shows that the space lag is also zero. The frictional pressure drop is given by

$$\Delta P = \frac{f l}{2D} G \langle u_g \rangle = \frac{f l}{2D} \left(\frac{W}{A_c} \right)^2 v_m \qquad \text{V1-32}$$

where the mean specific volume v_m in the heated duct is obtained from Eq. IV-77, thus

$$v_m = \frac{1}{\rho_m} = \frac{\langle u_f \rangle}{G} \qquad \text{V1-33}$$

If we insert in Eq. V1-32 the expression for the average velocity $\langle u_g \rangle$ given by Eq. IV-35 and since the space lag is zero, we can express the mean specific volume v_m as

$$v_m = v_f + \frac{1}{2} \frac{\Omega \ell}{G} \qquad \text{V1-34}$$

or in view of Eq. IV-21 as

$$v_m = v_f + \frac{1}{2} \left(\frac{dv}{di} \right)_p \frac{\dot{Q}}{W} \qquad \text{V1-35}$$

whence

$$\frac{dv_m}{dW} = - \frac{1}{2} \left(\frac{dv}{di} \right)_p \frac{\dot{Q}}{W^2} \qquad \text{V1-36}$$

Since Eq. V1-32 and Eq. V1-35 show that both ΔP and v_m are functions of W we can express the stability coefficient defined by Eq. V1-28 as

$$S = 2 + \frac{W}{v_m} \left(\frac{dv_m}{dW} \right)_{\dot{Q}} \qquad \text{V1-37}$$

whence from Eq. V1-36 we have

$$S = 2 - \frac{1}{2} \left(\frac{dv}{di} \right)_p \frac{\dot{Q}}{W} \frac{1}{v_m} \qquad \text{V1-38}$$

where (dv/di) is given by Eq. IV-15 or Eq. IV-8 depending on whether we are interested in the subcritical or in the supercritical region.

It can be seen from Eq. Vl-37 and Eq. Vl-30 that for a system where the mean specific volume does not depend on the mass flow rate the flow will be stable. For such incompressible flow system the coefficient of stability S has a value equal to 2. This is also the maximum value of S because when the fricition factor f in Eq. Vl-32 is a function of the Reynolds number then Eq. Vl-28 shows that S will have a value less than two. For example, for laminar flow it will have a value equal to unity.

For a boiling system at subcritical pressures or for a process of heating at supercritical pressures Eq. Vl-38 shows that the value of S can become negative because of the compressibility of the fluid. For such systems Eq. Vl-30 shows that the flow may become unstable.

In closing it should be emphasized that the density effect per se, can lead to excursive flow instabilities. Oscillatory flow instabilities results from a combined effect of time lag and of density variation. This will be analyzed in the two chapters that follow.

VII. Oscillatory Instability at Low Subcooling

VII.1 The Characteristic Equation and the Stability Map

In this chapter, and in the following one we shall investigate periodic, i.e., oscillatory flow phenomena. For this purpose we shall assume that the exponent s of the inlet velocity perturbation is given by $s = i\omega$ where the angular frequency ω, is a root of the characteristic equation, i.e., of Eq. V-15. In this chapter we shall consider the case of low subcooling, whereas, in the one that follows we shall consider the case of high subcooling.

For the case of low subcooling the characteristic equation, i.e., Eq. V-15, can be simplified by recalling that for low subcooling the time lag τ_b, given by Eq. III-19 will be short. Note, that the total transit time $\tau_3 - \tau_1$, which also appears in Eq. V-15 need not be short. This can be seen by considering Eq. IV-63, i.e.,

$$\tau_3 - \tau_1 = \frac{\Delta i_{21} \rho_f A_c}{q \xi} + \left(\frac{di}{dv}\right)_p \frac{A_c}{q \xi} \ln\left\{1 + \left(\frac{dv}{di}\right)_p \frac{\rho_f \dot{Q}}{W}\left(1 - \frac{W \Delta i_{21}}{\dot{Q}}\right)\right\} \qquad \text{VII-1}$$

which can be also expressed as:

$$\tau_3 - \tau_1 = \tau_b + \frac{1}{\Omega} \ln\left\{1 + \frac{\Omega \ell}{\bar{u}_1}\left(1 - \frac{\bar{\lambda}}{\ell}\right)\right\} \qquad \text{VII-2}$$

or as

$$\Omega(\tau_3 - \tau_1) = \Omega \tau_b + \ln \frac{\bar{u}_3}{\bar{u}_1} \qquad \text{VII-3}$$

Consequently, for short a space lag $\bar{\Lambda}$, and a short time lag τ_b, the transit time may be long for sufficiently long ducts and/or for low inlet velocities. It can be seen from Eq. VII-3 that the effect of time lag will be small if

$$\Omega \tau_b \ll 1 \qquad \text{VII-4}$$

which for subcritical pressures implies

$$\Omega \tau_b = \frac{\Delta v_{fg}}{v_f} \frac{\Delta i_{21}}{\Delta i_{fg}} < 1 \qquad \text{VII-5}$$

whereas, at supercritical pressure this inequality implies:

$$\Omega \tau_b = \frac{R}{P c_p} \frac{\Delta i_{21}}{v_f} < 1 \qquad \text{VII-6}$$

When the time lag $\bar{\Lambda}$ is short, then in Eq. V-15 the exponential term which contains τ_b, can be expanded and the characteristic equation reduces to

$$s^2 \{F_1 + F_2 - F_2 \Omega \tau_b\} + s\{F_3 + F_4 - \Omega(F_1 + F_2) - \Omega \tau_b (F_4 + F_7 - \Omega F_2)\} -$$
$$- \Omega\{F_3 + F_4 - F_5 - \Omega \tau_b (F_4 + F_7 - F_5)\} - \Omega F_6 e^{-s(\tau_3 - \tau_1)} = 0 \qquad \text{VII-7}$$

This equation can be cast in a dimensionless form by defining a dimensionless exponent

$$\gamma = s(\tau_3 - \tau_1) = s \Delta \tau \qquad \text{VII-8}$$

using this new variable, Eq. VII-7 can be expressed as

$$\zeta^2 + a\zeta + b + c\,e^{-\zeta} = 0 \qquad \text{VII-9}$$

where the dimensionless coefficients a, b and c are given by:

$$a = \Delta\tau\left\{\frac{F_3 + F_4 - \Omega(F_1+F_2) - \Omega\tau_b\,(F_4 + F_7 - \Omega F_2)}{F_1 + F_2 - F_2\,\Omega\tau_b}\right\} \qquad \text{VII-10}$$

$$b = -\Omega\Delta\tau^2\left\{\frac{F_3 + F_4 - F_5 - \Omega\tau_b\,(F_4 + F_7 - F_5)}{F_1 + F_2 - F_2\,\Omega\tau_b}\right\} \qquad \text{VII-11}$$

$$c = -\Omega\Delta\tau^2\left\{\frac{F_6}{F_1 + F_2 - F_2\,\Omega\tau_b}\right\} \qquad \text{VII-12}$$

and where the total transit time $\Delta\tau$ is given by Eq. VII-7. The coefficients a, b and c can be expressed also in terms of the pressure drops, thus

$$a = \frac{\Delta\tau}{M_f + M_g(1-\Omega\tau_b)}\left\{\frac{2\overline{\Delta P_{01}}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\frac{\partial\overline{\Delta P_{ex}}}{\partial u_1}\right| + \frac{\overline{\Delta P_a}}{u_{lm}} + \frac{2\overline{\Delta P_{23}}}{\langle u_g\rangle} + \frac{2\overline{\Delta P_{34}}}{\bar{u}_3} - \Omega\tau_b\left[\frac{\overline{\Delta P_a}}{\Omega(1-\bar{\lambda})} + \frac{\overline{\Delta P_{23}}}{\langle u_g\rangle} - \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3}\right]\right\} \qquad \text{VII-13}$$

$$b = -\frac{\Omega\Delta\tau^2}{M_f + M_g(1-\Omega\tau_b)}\left\{\frac{2\overline{\Delta P_{01}}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\frac{\partial\overline{\Delta P_{ex}}}{\partial u_1}\right| + \frac{\overline{\Delta P_{23}}}{\langle u_g\rangle} - \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} - \Omega\tau_b\left[\frac{\overline{\Delta P_a}}{\Omega(1-\bar{\lambda})} + \frac{\overline{\Delta P_{23}}}{\langle u_g\rangle} - \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3}\right]\right\} \qquad \text{VII-14}$$

$$c = -\frac{\Omega\Delta\tau^2}{M_f + M_g(1-\Omega\tau_b)}\left\{\frac{\overline{\Delta P_a}}{\bar{u}_1} + \frac{\overline{\Delta P_{bg}}}{\bar{u}_1} + \frac{\overline{\Delta P_{23}}}{\bar{u}_1} + \frac{\overline{\Delta P_{34}}}{u_1}\right\} \qquad \text{VII-15}$$

Equation VII-9 is a second order exponential polynomial with one time delay. Stability maps for such polynomials have been recently presented by Bhatt and Hsu [65, 64]. One such map is shown in Figure VII-1, it is in the c-b plane with the coefficient "a" as a parameter. The lines for which the coefficient "a" is constant are stability boundary curves. For example, for given values of the coefficients "b" and "a", the stable region of variation for the coefficient "c" is shown by the line segment AB. The segment CD is another stable range for constant values of "b" and of "a".

Figure VII-1 is the stability map which can be used to differentiate the regions of stable operation from the region of unstable, i.e., of oscillatory flow in the heated duct. However, because of the complicated nature of the coefficients "a", "b", and "c" which appear on this map, it is rather difficult to discuss and analyze the effects of the various parameters. It is desirable, therefore, to simplify the characteristic equation in order to obtain simple stability criteria. This will be done in the section that follows by neglecting the inertia terms in Eq. VII-7.

VII.2 Stability Criterion for the Case of Small Inertia

VII.2 The Characteristic Equation

If we neglect the inertia terms F_1 and F_2 in Eq. VII-7, the characteristic equation reduces to its simplest form given by

$$s + A + B e^{-s\Delta\tau} = 0 \qquad \text{VII-16}$$

where the coefficients A and B are given by

$$A = -\Omega \left\{ \frac{F_3 + F_4 - F_5 - \Omega\tau_L (F_4 + F_7 - F_r)}{F_3 + F_4 - \Omega\tau_L (F_4 + F_7)} \right\} \qquad \text{VII-17}$$

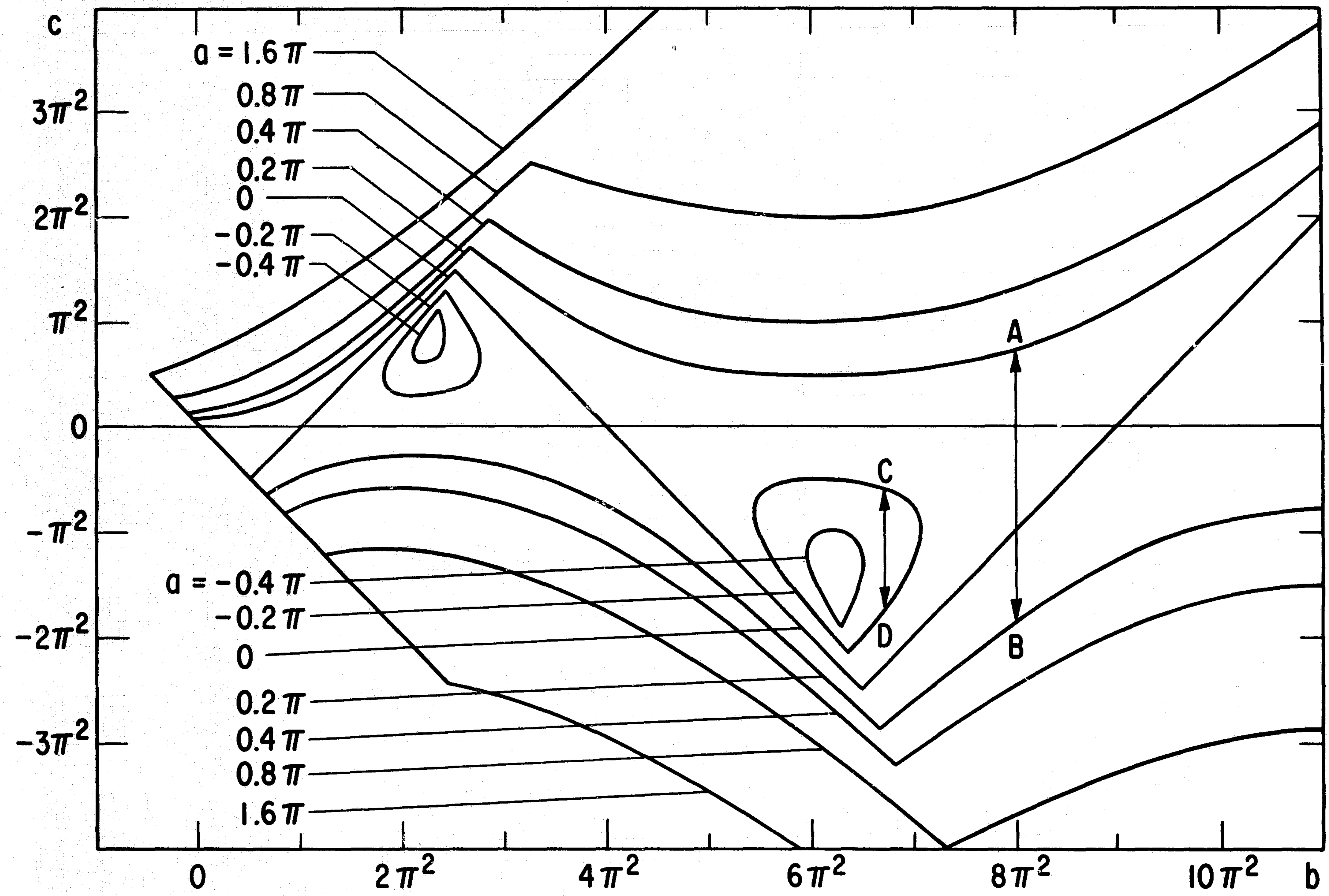

FIGURE VII-1 THE BHATT-HSU STABILITY MAP FOR THE SECOND ORDER EXPONENTIAL POLYNOMIAL: $\zeta^2 + a\zeta + b + ce^{-\zeta} = 0$

and
$$\mathcal{B} = -\Omega \frac{F_6}{F_3 + F_4 - \Omega \tau_b (F_4 + F_1)} \qquad \text{VII-18}$$

It is important to note that a characteristic equation of the form of a first order exponential polynomial with one time delay describes the onset of "chugging" combustion instabilities as shown by Crocco and Cheng [48]. Since Eq. VII-16 is of such a form, we can use the results of Crocco and Cheng [48] to analyze the flow stability in this problem. The difference between the present problem and that of combustion is the physical meaning of the coefficients A and B. In this problem they depend on various pressure drops in the system which were obtained from the momentum equation. In the combustion problem the coefficients are obtained from the continuity equation and depend, among others, on the process of combustion.

We note also that the results of Stenning [62] can be expressed in terms of a characteristic equation of the form of a first order exponential polynomial with one time delay. However, since Stenning [62] did not formulate his analysis of boiling instabilities in terms of the momentum equation,* the coefficients in his characteristic equations do not depend upon the pressure drops.

VII.2.2 Unconditional Flow Stability

It was shown by Crocco and Cheng [48] that no matter what the value of the time delay $\Delta\tau$ may be the flow will be unconditionally stable if the coefficients A and B in Eq. VII-16 satisfy the following inequality

$$\frac{A}{B} > 1 \qquad \text{VII-19}$$

* The problem was formulated in terms of the continuity and of the energy equation.

Because of its importance, we shall define this ratio as the Stability Number N_s. In view of Eq. VII-17 and VII-18, it can be expressed as

$$N_s = \frac{F_3 + F_4 - F_5 - \Omega \tau_b (F_4 + F_7 - F_5)}{F_6} > 1 \qquad \text{VII-20}$$

This stability criterion can be put also in the form of

$$F_3 + F_4 - F_5 - \Omega \tau_b (F_4 + F_7 - F_5) - F_6 > 0 \qquad \text{VII-21}$$

whence upon inserting the values for the influence coefficients in Eq. VII-27 we obtain the inequality which must be satisfied for unconditional flow stability, thus

$$\frac{2\Delta P_{01}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\frac{\partial \overline{\Delta P_{ex}}}{\partial u_1}\right| - \qquad \text{VII-22}$$

$$- \frac{\overline{\Delta P_{23}}}{\bar{u}_1}\left(1 - \frac{\bar{u}_1}{\langle u_g \rangle}\right) - \frac{\overline{\Delta P_{bg}}}{\bar{u}_1}\left(1 + \frac{\bar{u}_1}{u_m}\right) - \frac{\overline{\Delta P_{34}}}{\bar{u}_1}\left(1 - \frac{\bar{u}_1}{\bar{u}_3}\right) -$$

$$- \Omega \tau_b \left(\frac{\overline{\Delta P_a}}{\Omega (1-\bar{\lambda})} + \frac{\overline{\Delta P_{23}}}{\langle u_g \rangle} - \frac{\overline{\Delta P_{bg}}}{u_m} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \right) > 0$$

This criterion clearly indicates the destabilizing effect of the pressure drops in the "light" fluid region and the stabilizing effect of the pressure drops in the "heavy" fluid region.

For once-through systems, when the acceleration and the gravitational terms can be neglected, Eq. VII-22 reduces to

$$\frac{2\overline{\Delta P_{01}}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \frac{\partial \overline{\Delta P_{ex}}}{\partial u_1} - \frac{\overline{\Delta P_{23}}}{\bar{u}_1}\left(1 - \frac{\bar{u}_1}{\langle u_g \rangle}\right) - \frac{\overline{\Delta P_{34}}}{\bar{u}_1}\left(1 - \frac{\bar{u}_1}{\bar{u}_3}\right) - \qquad \text{VII-23}$$

$$- \Omega \tau_b \left[\frac{\overline{\Delta P_{23}}}{\langle u_g \rangle} + \frac{\overline{\Delta P_{34}}}{u_3} \right] > 0$$

If we now approximate $\langle u_4 \rangle$ by $\overline{u_3}$ we can express Eq. VII-23 as

$$\frac{2\overline{\Delta P_{01}}}{\overline{u_1}} + \frac{2\overline{\Delta P_{12}}}{\overline{u_1}} + \left| \frac{\partial \overline{\Delta P_{ex}}}{\partial u_1} \right| - \frac{\overline{\Delta P_{23}} + \overline{\Delta P_{34}}}{\overline{u_1}} \; \frac{\overline{u_3} - \overline{u_1}}{\overline{u_3}} - \frac{\Omega \tau_b}{\overline{u_3}} \left(\overline{\Delta P_{23}} + \overline{\Delta P_{34}} \right) > 0 \qquad \text{VII-24}$$

Defining by $\overline{\Delta P_f}$ the sum of the pressure drops in the "heavy" fluid region.

$$\overline{\Delta P_f} = \overline{\Delta P_{01}} + \overline{\Delta P_{12}} + \overline{\Delta P_{ex}} \qquad \text{VII-25}$$

and by $\overline{\Delta P_g}$ the sum of the frictional and of the exit pressure drops in the "light" fluid region

$$\overline{\Delta P_g} = \overline{\Delta P_{23}} + \overline{\Delta P_{34}} \qquad \text{VII-26}$$

we can express Eq. VII-24 as

$$\frac{2\overline{\Delta P_f}}{\overline{\Delta P_g}} \; \frac{\overline{u_3}}{\overline{u_3} - \overline{u_1}} - \Omega \tau_b \; \frac{u_1}{\overline{u_3} - \overline{u_1}} > 1 \qquad \text{VII-27}$$

whence

$$\frac{2\overline{\Delta P_f}}{\overline{\Delta P_g}} \; \frac{u_1 + \Omega(1-\overline{\lambda})}{\Omega(1-\overline{\lambda})} - \frac{\Omega \tau_b \overline{u_1}}{\Omega(1-\overline{\lambda})} > 1 \qquad \text{VII-28}$$

Inserting now the expressions for the characteristic reaction frequency Ω, for the time lag τ_b, given by Eq. IV-23 and III-19, respectively, we obtain

$$\frac{2\overline{\Delta P_f}}{\overline{\Delta P_g}} \left(\frac{di}{dv} \right) \frac{A_c}{q\xi} \; \frac{\overline{u_1}}{(1-\overline{\lambda})} \left[1 + \left(\frac{dv}{di} \right) \frac{q\xi}{A_c} \; \frac{(1-\overline{\lambda})}{\overline{u_1}} \right] - \frac{\overline{u_1}}{1-\overline{\lambda}} \; \frac{\rho_f A_c \Delta i_{21}}{q\xi} > 1 \qquad \text{VII-29}$$

This inequality can be expressed also in terms of the total mass flow rate and of the total heat flow $\dot{Q}$. Thus

$$\frac{2\overline{\Delta P_f}}{\overline{\Delta P_g}}\left(\frac{di}{dv}\right)\frac{1}{\rho_f}\frac{W}{\dot{Q}\left(1-\frac{W\Delta i_{21}}{Q}\right)}\left[1+\left(\frac{dv}{di}\right)\frac{\rho_f\dot{Q}}{W}\left(1-\frac{W\Delta i_{21}}{Q}\right)\right]-\frac{W\Delta i_{21}}{\dot{Q}\left(1-\frac{W\Delta i_{21}}{Q}\right)}>1 \qquad \text{VII-30}$$

Again, we differentiate the process of boiling at subcritical pressure from the process of heating at supercritical pressures by using the appropriate equation of state, thus at subcritical pressure we use Eq. IV-15, i.e.,

$$\left(\frac{dv}{di}\right)_p = \frac{\Delta v_{fg}}{\Delta i_{fg}} \qquad \text{VII-31}$$

whereas at supercritical pressures we use Eq. IV-8, i.e.,

$$\left(\frac{dv}{di}\right)_p = \frac{R}{P c_p} \qquad \text{VII-32}$$

The implication of Eq. VII-30 will be discussed in Section VII.3.

VII.2.3 Conditional Stability

Following again Crocco and Cheng [48] we can determine the relation between the critical transit time $\Delta\tau_c$ and the critical frequencies ω_c correspondi[ng] to neutral oscillations. Such a relation is obtained by separating the real and imaginary parts of Eq. VII-16, thus

$$i\omega + A + B[\cos\omega_c\Delta\tau_c - i\sin\omega_c\Delta\tau_c] = 0 \qquad \text{VII-33}$$

whence

$$\sin\omega_c\Delta\tau_c = -\frac{\omega_c}{B} \qquad \text{VII-34}$$

and

$$\cos \omega_c \Delta\tau_c = -\frac{A}{B} \qquad \text{VII-35}$$

where the coefficient B, given by Eq. VII-18, can be expressed in terms of the pressure drops thus

$$B = -\frac{\Omega F_6}{F_3 + F_4 - \Omega\tau_b\,(F_4 + F_7)} \qquad \text{VII-36}$$

where

$$F_6 = \frac{\overline{\Delta P_a}}{\bar{u}_1} + \frac{\overline{\Delta P_{b9}}}{\bar{u}_1} + \frac{\overline{\Delta P_{23}}}{\bar{u}_1} + \frac{\overline{\Delta P_{34}}}{\bar{u}_1} \qquad \text{VII-37}$$

and

$$F_3 + F_4 - \Omega\tau_b\,(F_4 + F_7) = \frac{2\overline{\Delta P_{a}}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\frac{\partial \overline{\Delta P_{ex}}}{\partial u_1}\right| + \frac{\overline{\Delta P_4}}{u_{lm}} + \frac{2\overline{\Delta P_{23}}}{\langle u_g\rangle} + \frac{2\overline{\Delta P_{34}}}{\bar{u}_3} -$$

$$- \Omega\tau_b\left[\frac{\overline{\Delta P_g}}{u_{lm}} + \frac{\overline{\Delta P_a}}{\Omega(l-\bar{\lambda})} + \frac{2\overline{\Delta P_{23}}}{\langle u_g\rangle} + \frac{2\overline{\Delta P_{34}}}{\bar{u}_3}\right] \qquad \text{VII-38}$$

The stability number N_s, given by Eq. VII-20, becomes when expressed in terms of the pressure drops:

$$N_s = \frac{\dfrac{2\overline{\Delta P_{01}}}{\bar{u}_1} + \dfrac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\dfrac{\partial \overline{\Delta P_{ex}}}{\partial u_1}\right| + \left\{\dfrac{\overline{\Delta P_{23}}}{\langle u_g\rangle} - \dfrac{\overline{\Delta P_{b9}}}{u_m} + \dfrac{\overline{\Delta P_{34}}}{\bar{u}_3}\right\}(1-\Omega\tau_b) - \dfrac{\Omega\tau_b\,\Delta P_a}{\Omega(l-\bar{\lambda})}}{\dfrac{\overline{\Delta P_a}}{\bar{u}_1} + \dfrac{\overline{\Delta P_{b9}}}{\bar{u}_1} + \dfrac{\overline{\Delta P_{23}}}{\bar{u}_1} + \dfrac{\overline{\Delta P_{34}}}{\bar{u}_1}} \qquad \text{VII-39}$$

The critical frequency ω_c, is obtained from Equation VII-34 and Equation VII-35, thus

$$\omega_c = \sqrt{B^2 - A^2} \qquad \text{VII-40}$$

whereas, the critical transit time $\Delta\tau$ is given by Eq. VII-35 and Eq. VII-40, thus

$$(\tau_3 - \tau_1)_c = \Delta\tau_c = \frac{1}{\sqrt{B^2 - A^2}} \left\{ \pi - \cos^{-1}\left(\frac{A}{B}\right) \right\} \qquad \text{VII-41}$$

As discussed by Crocco and Cheng [48] if the inequality given by Eq. VII-19 is not satisfied, then stability is still possible if the angular frequency of the perturbation and the transit time satisfy the following inequalities

$$\omega > \omega_c \qquad \text{VII-42}$$

and

$$\Delta\tau < \Delta\tau_c \qquad \text{VII-43}$$

The system is intrinsically unstable if the directions of the inequalities are reversed. Furthermore, when

$$\Delta\tau = \Delta\tau_c \qquad \text{VII-44}$$

then Eq. VII-16 has an oscillatory solution with an angular frequency ω_c.

The preceding results can be plotted against the stability number N_s, given by Eq. VII-20, i.e., by Eq. VII-34. For this purpose we shall define also the period of the oscillation by

$$T = \frac{2\pi}{\omega_c} \qquad \text{VII-45}$$

We can form now the ratio of the critical transit time to the period and express it as function of the stability number N_s, thus from Eq. VII-45 and Eq. VII-41 we obtain

$$\frac{\Delta\tau_c}{T} = \frac{\omega_c(\tau_3-\tau_1)_c}{2\pi} = \frac{1}{2} - \frac{1}{2\pi}\cos^{-1} N_s \qquad \text{VII-46}$$

The critical angular frequency can be also expressed as functions of N_s, thus from Eq. VII-40,

$$\frac{\omega_c}{|B|} = \sqrt{1-N_s^2} \qquad \text{VII-47}$$

Similarly, by means of Eq. VII-41 we can express the critical transit time as function of N_s, thus

$$\frac{\Delta\tau_c}{2\pi|B|} = \frac{1}{\sqrt{1-N_s^2}}\left\{\frac{1}{2} - \frac{1}{2\pi}\cos^{-1} N_s\right\} \qquad \text{VII-48}$$

Eq. VII-48, VII-47, and VII-46 are plotted versus the stability number N_s, in Figure VII-2. The significance of this map is discussed in the following section.

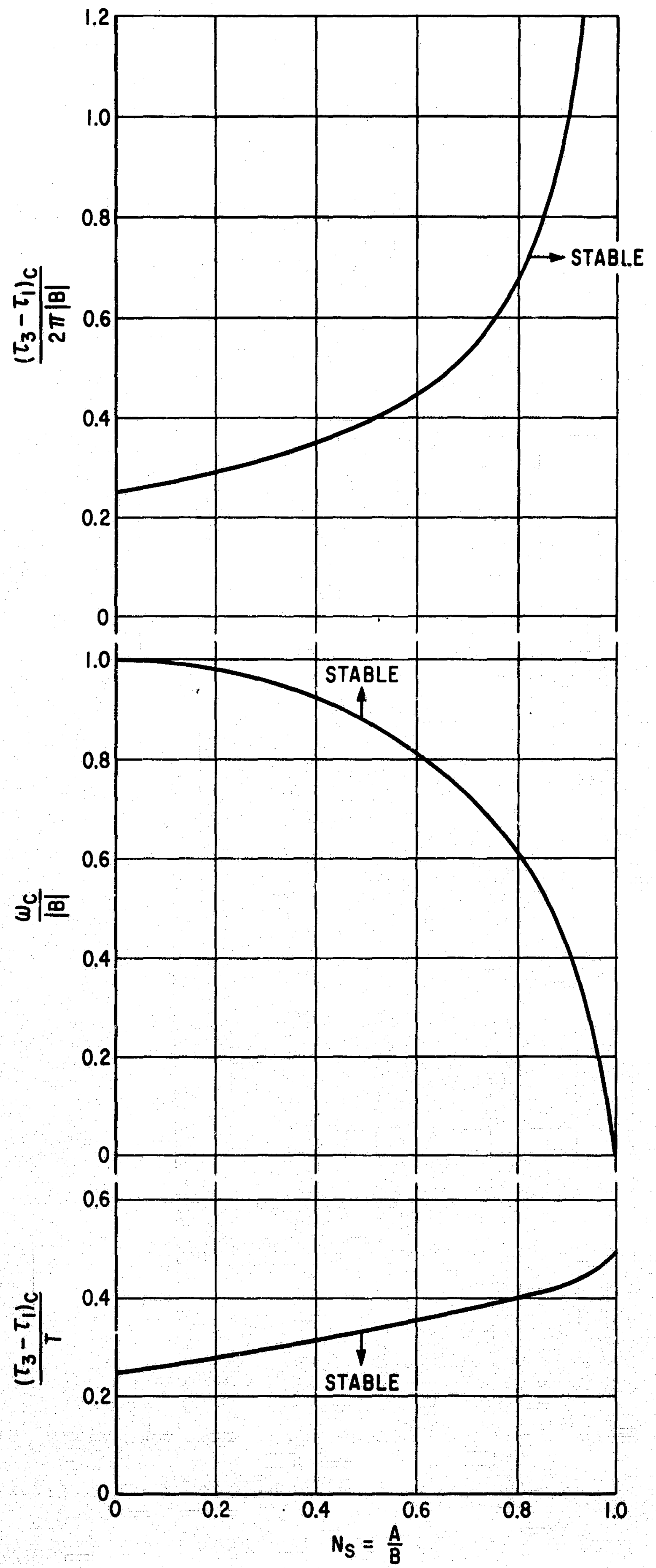

FIGURE VII - 2 THE STABILITY MAP FOR THE FIRST ORDER EXPONENTIAL POLYNOMIAL: $s + A + Be^{-s\Delta\tau} = 0$

VII.3 Effects of Various Parameters and Methods for Improving Flow Stability

The effects which various parameters have on the propensity to induce flow oscillation at low subcooling can be evaluated by examining Eq. VII-22 or Eq. VII-30. It can be seen that the variation of any parameter that tends to decrease the positive value of the left hand side of these equations has a destabilizing effect. For example, increasing the various pressure drop terms in the "light" fluid region has a destabilizing effect. Similarly, an increase of subcooling tends to destabilize the flow. Vice versa, an increase of the inlet pressure drop or a change of the pump characteristics will stabilize the flow.

Although the preceding results have not yet been tested against experimental data, the form of the simplified stability criterion given by Eq. VII-29, seems to be correct. This statement is based on a comparison of Eq. VII-29 with the empirical criterion for predicting boiling instabilities recently proposed by Serov and Smirnov (66). In the nomenclature of this paper, their criterion is given by

$$\frac{\overline{\Delta P}_{01}+\overline{\Delta P}_{12}}{\overline{\Delta P}_{23}+\overline{\Delta P}_{34}} > a\left\{\frac{q(1-\Lambda)}{D\,\bar{u}_1}\,\frac{\Delta v_{fg}}{\Delta i_{fg}}\right\} - b\left\{\rho_f\,\bar{u}_1\,V_0\,\rho_g\,dv_g/dP_0\right\} \qquad \text{VII-49}$$

where a and b are two constants to be determined from experiments, D is the diameter of the pipe; V_0 is the volume occupied by the steam and (dv_g/dP) is the variation of the specific volume of the steam with pressure. Consequently, the second term on the right hand side of Eq. VII-49 represents the effect of compressibility. This effect was neglected in the present analysis.

It was reported by Serov and Smirnov [66] that Eq. VII-49 was successful in correlating data and predicting the onset of flow instabilities in boiling of water at pressure of 30, 50, 70 and 100 atmospheres.

If we neglect the effects of compressibility in Eq. VII-49 and compare it to Eq. VII-29 and VII-31, it can be seen that Eq. VII-49 is incorporated in Eq. VII-29. We note also that this latter equation is a simplified form of Eq. VII-23; i.e. of Eq. VII-9 which are therefore more general and complete.

Further experimental evidence that gives support to the form of Eq. IV-29 is shown on Figure VII-3 which is reproduced from the paper by Platt and Wood [10]. It can be seen from this figure that either increasing the power input and/or decreasing the mass flow rate has a destabilizing effect. The same results are predicted by Eq. IV-29.

Perhaps the result of greatest significance revealed in the present investigation is the similarity between the characteristic equations for predicting "chugging" combustion oscillations and the characteristic equation for predicting low frequency flow oscillations in heated ducts at near critical and at super-critical pressures. Since it is well known (see for example [48]) that "chugging" combustion instabilities can be stabilized by an appropriate servo-control mechanism, the results of this investigation indicate that low frequency flow oscillation at near critical and at supercritical pressures may be also stabilized. This important conclusion is demonstrated on Figure VII-2 which shows also the effect of various parameters on the propensity toward oscillatory flow.

It can be seen on Figure VII-2 that even when the stability number N_S is less than unity, the flow may be stable if the frequency of the inlet perturbation is higher than the critical frequency ω_c. Similarly, the flow can be stable if the total transit time is shorter than the critical one. The values of ω_c and of $(\tau_3-\tau_1)_c = \Delta\tau_c$ are plotted in Figure VIII-2 in terms of the stability number N_S and of the coefficient B given by Eq. VII-39 Eq. VII-36 respectively.

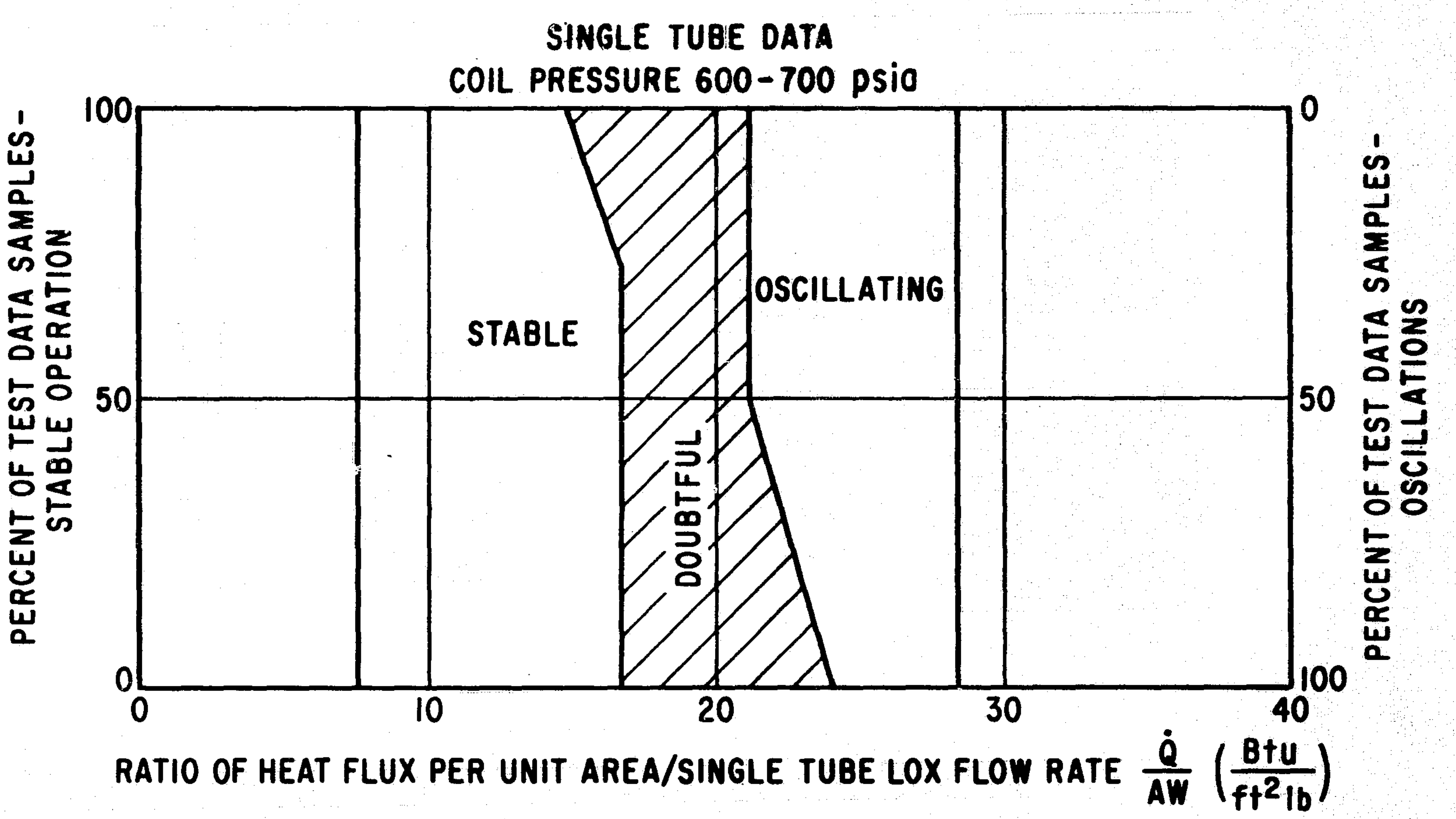

FIGURE VII-3 PERCENTAGE OF HEAT EXCHANGER TEST DATA SAMPLES SHOWING STEADY OPERATION VS. HEAT FLUX PER UNIT AREA PER UNIT MASS FLOW RATE, REPORTED BY PLATT AND WOOD.

The effects which the variations of the various parameters have on the flow stability can be evaluated from Figure VII-2 by considering whether the variation results in an increase of the stable region. For example, it can be seen from Figure VII-2 that for a constant value of N_s an increase of the delay time has a destabilizing effect because for sufficiently long delays $\Delta\tau$ will become larger than $\Delta\tau_c$. We note that this quantitative conclusion is in agreement with the qualitative description of the destabilizing effect of the time delay presented in Section II-4. It can be also seen from Figure VII-2 that increasing the frequency of the inlet perturbation at a constant value of N_s, has a stabilizing effect because for sufficiently high frequency ω will become larger than ω_c. Furthermore, Figure VII-2 shows that an unstable flow; i.e., a flow for which $\omega < \omega_c$ and $\Delta\tau > \Delta\tau_c$ can be stabilized by increasing the value of the stability number N_s.

We close this section by observing that the foregoing conclusions and results are new and have not yet been verified against experimental data. If confirmed, then the results of this study provides a method whereby stable operation can be insured on an intrinsically unstable region.

VIII. Oscillatory Instability at High Subcooling

VIII.1 The Characteristic Equation and the Stability Criterion

We shall consider now the case of high inlet subcooling which implies a long time lag τ_b and a long space lag $\bar{\Lambda}$. For such system Eq. VII-3 indicates that the transit time and the time lag will be of the same order of magnitude. Since both time delays are long, we shall neglect the exponential terms in the characteristic equation given by Eq. V-15, which reduces then to

$$s^2\{F_1+F_2\} + s\{F_3+F_4-\Omega(F_1+F_2)\} - \Omega\{F_3+F_4-F_5\} - \qquad \text{VIII-1}$$

$$-\frac{\Omega}{s}\{s^2F_2 + s[F_4+F_7-\Omega F_2] - \Omega[F_4+F_7-F_5]\} = 0$$

which can be rearranged and expressed as

$$s^3 + s^2\left\{\frac{F_3+F_4-\Omega(F_1+F_2)-\Omega F_2}{F_1+F_2}\right\} - \qquad \text{VIII-2}$$

$$-s\left\{\frac{\Omega[F_3+F_4-F_5-(F_4+F_7-\Omega F_2)}{F_1+F_2}\right\} + \Omega^2\left\{\frac{F_4+F_7-F_5}{F_1+F_2}\right\} = 0$$

where the sums of the influence coefficients are related to the pressure drops by the following relations

$$F_1+F_2 = M_f + M_g \qquad \text{VIII-3}$$

$$F_3 + F_4 - \Omega(F_1 + F_2) - \Omega F_2 = \frac{2\overline{\Delta P_{01}}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\frac{\partial \overline{\Delta P_{ex}}}{\partial u_1}\right| + \frac{\overline{\Delta P_a}}{u_{em}} +$$

$$+ \frac{2\overline{\Delta P_{23}}}{\langle u_4 \rangle} + \frac{2\overline{\Delta P_{34}}}{\bar{u}_3} - \left(\frac{dv}{di}\right)_p \frac{q\xi}{A_c}(M_f + M_4) - \left(\frac{dv}{di}\right)_p \frac{q\xi}{A_c} M_4 \qquad \text{VIII-4}$$

$$F_3 + F_4 - F_5 - (F_4 + F_7 - \Omega F_2) = \frac{2\overline{\Delta P_{01}}}{\bar{u}_1} + \frac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\frac{\partial \overline{\Delta P_{ex}}}{\partial u_1}\right| - \qquad \text{VIII-5}$$

$$- \left[\frac{\Delta P_a}{u_{em}} + \frac{\overline{\Delta P_a}}{\Omega(1-\bar{\lambda})} + \frac{2\overline{\Delta P_{23}}}{\langle u_4 \rangle} + \frac{2\overline{\Delta P_{34}}}{\bar{u}_3}\right] + \left(\frac{dv}{di}\right)_p \frac{q\xi}{A_c} M_4$$

$$F_4 + F_7 - F_5 = \frac{\overline{\Delta P_a}}{\Omega(1-\bar{\lambda})} + \frac{\overline{\Delta P_{23}}}{\langle u_4 \rangle} - \frac{\overline{\Delta P_{b4}}}{u_m} + \frac{\overline{\Delta P_{34}}}{\bar{u}_3} \qquad \text{VIII-6}$$

It can be seen that the characteristic equation is a cubic equation of the form of

$$S^3 + S^2\left\{\frac{b}{a}\right\} + S\left\{\frac{c}{a}\right\} + \left\{\frac{d}{a}\right\} = 0 \qquad \text{VIII-7}$$

where the coefficient a, b, c and d are given by the corresponding terms of Eq. VII-2.

The problem of determining the conditions for neutral stability is solved again by substituting S = $i\omega$ in Eq. VII-7.

Thus VIII-8

$$-i\omega^3 - \omega^2\left\{\frac{b}{a}\right\} - i\omega\left\{\frac{c}{a}\right\} + \left\{\frac{d}{a}\right\} = 0$$

whence upon separating the real and the imaginary parts we have

$$\omega^2 = -\frac{c}{a} \qquad \text{VIII-9}$$

and

$$\omega^2 = \frac{d}{b} \qquad \text{VIII-10}$$

Consequently for oscillations to be possible the coefficients a,b,c and d in Eq. VII-9 and Eq. VII-10 must satisfy the following relation:

$$-\frac{c}{a} = \frac{d}{b} \qquad \text{VIII-11}$$

whence, the values of the influence coefficients must be such as to satisfy the following expression:

$$-\frac{F_3 + F_4 - F_5 - (F_1 + F_7 - \Omega F_2)}{\Omega (F_1 + F_2)} = \frac{F_4 + F_7 - F_5}{F_3 + F_4 - \Omega (F_1 + F_2) - \Omega F_2} \qquad \text{VIII-12}$$

It can be seen from Eq. VIII-4 and Eq. VIII-6 that, unless the effects of inertia or of gravity become dominant, the right hand side of Eq. VIII-12 is a positive quantity. Consequently, Eq. VIII-12 indicates that oscillation can occur only if

$$F_3 + F_4 - F_5 - (F_4 + F_7 - \Omega F_2) < 0 \qquad \text{VIII-13}$$

Therefore, the flow will be stable if

$$F_3 + F_4 - F_5 - (F_4 + F_7 - \Omega F_2) > 0 \qquad \text{VIII-14}$$

In view of Eq. VIII-5, this inequality can be expressed also as:

$$\frac{\dfrac{2\Delta P_{01}}{\bar{u}_1} + \dfrac{2\overline{\Delta P_{12}}}{\bar{u}_1} + \left|\dfrac{\partial \Delta P_{ex}}{\partial u_1}\right| + \Omega F_2}{\dfrac{\overline{\Delta P_a}}{u_{em}} + \dfrac{\overline{\Delta P_a}}{\Omega(1-\bar{\lambda})} + \dfrac{\Delta P_{b4}}{u_m} + \dfrac{\Delta P_{23}}{\langle u_4 \rangle} + \dfrac{\Delta P_{34}}{\bar{u}_3}} > 1 \qquad \text{VIII-15}$$

For oscillatory flow, Eq. VIII-13 and Eq. VIII-9 indicate that the angular frequency will be given by

$$\omega^2 = \left| - \frac{c}{a} \right| = \frac{d}{b} \qquad \text{VIII-16}$$

which, when expressed in terms of the influence coefficients, becomes

$$\omega = \left[\frac{F_4 + F_7 - \Omega F_2 - (F_3 + F_4 - F_5)}{\Omega (F_1 + F_2)} \right]^{1/2} \qquad \text{VIII-17}$$

It should be noted, again, that the values of these influence coefficients should satisfy Eq. VIII-16, i.e. Eq. VIII-12.

VIII.2 Effects of Various Parameters and Methods for Improving Flow Stability

The effects of the various parameters can be evaluated by examining the inequality given by Eq. VIII-15. It can be seen that any variation which tends to increase the value of the left hand side of this equation will have a stabilizing effect. Thus, the flow can be stabilized by increasing the pressure drops in the "heavy" fluid region, whereas it will be destabilized by increasing the pressure drops in the "light" fluid region.

The effect of subcooling can be evaluated by comparing Eq. VIII-14 and Eq. VIII-15 with Eq. VII-20 and Eq. VII-39. Since the velocities in the "light" fluid region are higher than the inlet velocity it can be seen from such a comparison that the inequality applicable at high subcoolings, i.e. Eq. VIII-14 is less restrictive than that corresponding to low subcoolings, i.e., than Eq. VII-20. Consequently, the flow is more stable at high subcoolings. However, since Eq. VII-20 indicates also that an increase in subcooling destabilizes the flow, we conclude that this destabilizing effect must go through a maximum at intermediate subcoolings. For boiling systems, this conclusion is in agreement with the experimental results of Gouse (67) who was apparently the first to notice this effect. At super critical pressures, experimental data, which could be used to test this conclusion, are not yet available.

IX. DISCUSSION

The instability mechanism investigated in this paper was based on the destabilizing effects of time lags and of density variations in the heated duct.* It was shown that, in the near critical and in the supercritical region, these destabilizing effects can induce flow excursions as well as flow oscillations.

The characteristic equation, i.e., Eq. V-15, which predicts the onset of these instabilities is given by a third order exponential polynomial with two time delays. Because of its complex nature this equation was not solved at this time. Instead, simplified stability criteria were sought and derived by assuming that the inlet subcooling was either low or high. This approach seemed preferable for several reasons.

First, the simple stability criteria are more instructive and helpful for gaining an understanding of the essential nature of the instability.

Two, the result shows that the dominance of a particular parameter results in a particular angular frequency of oscillations (see Eq. VII-40 and VIII-17). Consequently, the cause of instability can be determined from a trace of the flow oscillation.

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*Other mechanisms which may induce flow oscillation were discussed in Section II-7.

Finally, simplified stability criteria such as Eq. VI-20, VII-22, VII-42 and VIII-15 are more amenable to a qualitative study of the effects which variations of the various parameters may have on inducing or on preventing flow excursions and/or flow oscillations. Indeed, only if the results from such a study are in agreement with experimental observations, a detailed quantitative solution of the more complicated characteristic equation can be justified.

It was discussed in Sections VI-2, VII-3 and VIII-2 that the predictions based on the simplified stability criteria are indeed in qualitative agreement with the experimental data. This agreement warrants therefore a more complete study of the characteristic equation together with a quantitative comparison with the experimental data.

Last but not least the simple criteria are most useful in indicating the improvements and changes in the design or in the operation of the system which would insure stable flow. Several such improvements were discussed in Sections VI-2, VII-3 and VIII-2. It was noted there that the results of this study indicate that low frequency thermally induced flow oscillations in the near critical and in the supercritical pressure region, could be stabilized by an appropriate servo-control mechanism. Whether this important conclusion is indeed correct remains to be shown by future experiments.

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Appendix A

The Near-Critical Thermodynamic Region

The success of an investigation concerned with predicting or interpreting the behaviour of a thermo-dydraulic system depends on the availability and on the accuracy of data giving the values of thermodynamic and transport properties of the fluid in the region of interest. It is the purpose of this appendix to summarize, briefly, the status of present understanding of thermodynamic phenomena that take place in a region near the critical thermodynamic point. For additional discussion, the reader is referred to the extensive reviews by Rice (A1) and by Hammell (A2).

Consider a fluid at a pressure slightly above the critical pressure flowing through a heat exchanger. If the temperature of the fluid at the entrance is considerably below the critical temperature, i.e., $T \ll T_c$, the fluid will have a density close to that of a liquid whereas at the exit, if the fluid temperature is considerably above T_c, the density will approximate that of a perfect gas. Consequently, in passing through the heat exchanger the fluid will undergo a change of properties from a liquid-like fluid at the entrance to a gas-like fluid at the exit. Since the properties of the fluid will affect the performance of the system is becomes necessary first to examine the nature of this change and then to express it quantitatively.

At subcritical pressures the presence of two phases is distinguished by a difference in density and by the existence of an interface between the phases. At supercritical pressures such a distinction cannot be made because at these pressures as well as at the critical one the interface, the heat of vaporization, as well as the surface energy, all vanish.

There is no general agreement as to the structure of the medium and of the mechanism of phase transition in the critical and in the supercritical region. Different explanations and descriptions are advanced by different authors.

Some authors like Rosen (A3) and Semenchenko (A4) analyze the thermodynamic characteristics of a medium in the supercritical region by assuming an equation of state like the Van der Waals' or the Dieterici equations.

Hirschfelder, Curtis and Bird (A5) describe the fluid in the neighborhood of the critical point as consisting of a large number of clusters of molecules of various sizes. The system can be idealized by assuming that the density can be described by a distribution function which has for its two limits the densities of the two phases. The fluctuation in density, which can be expected from the theory of fluctuations, becomes very large in the vicinity of the critical point. These large fluctuations and the formation of molecular clusters in the neighborhood of this point result in a large increase of the specific heat at constant volume.

Mayer and co-workers (A6) propose a theory of condensation based on the cluster theory of imperfect gases from which they predict the existence of an anomalous region above the temperature of the usually observed critical point. This region extends up to the highest isotherm for which $(\partial P/\partial v)_T$, has anywhere a zero value. In this region, isotherms exist having no variation in pressure over a finite density range, but having at all densities continuous derivatives with respect to pressure. Various aspects of this theory are discussed further in (A5).

A great number of authors distinguish two phases in the supercritical region: a heavy, liquid-like phase and a light, gas-like phase. The difference between their results stems from the different approaches used to locate the boundary between the two phases and from the different descriptions of the characteristics of the phase transition.

In a preceeding section we have discussed already Goldman's (A7-A8) descriptions of the supercritical region and of the similarity between the heat transfer and flow processes at supercritical pressure and those that take place at subcritical pressure during the process of boiling. However, Goldman did not formulate, quantitatively, the problem nor did he say how and where to locate the boundary or the region between the liquid-like and the gas-like phase.

Following Goldman, Hendricks et al (A9) consider "boiling-like" phenomena at supercritical pressures and, in analogy with boiling, they introduce a specific volume for the fluid of the form of Eq. A1.

$$v_m = v_f + \frac{X}{2} \left(v_g - v_f \right) \qquad \text{(A-1)}$$

In place of the quality they introduce a weighting function for the heavy and light species. However, no reference is made in their paper as to how to determine, quantitatively, this distribution function.

In numerous textbooks (A-10) among others, the boundary between the liquid and the gas in the supercritical region is taken to be the critical isotherm. Other authors like Thiesen (A-11), Trautz and Ader (A-12) among others take the critical isochor for this boundary and consider it as the extension of the saturation line into the supercritical region.

In the subcritical region various thermodynamic properties such as the specific heat, the compressibility, the coefficient of thermal expansion and others change discontinuously or reach a maximum value at the coexistence, i.e., the saturation line. This line can be therefore looked upon as the locus of points for these discontinuities or maxima. Consequently, numerous authors consider the extension of the saturation line into the supercritical region to be the line which is the locus of points where the thermodynamic properties listed below reach a maximum:

$$\frac{\partial^2 i}{\partial T \partial P} = \left(\frac{\partial c_p}{\partial P}\right)_T = \left(\frac{\partial^2 v}{\partial T^2}\right)_T = 0 \tag{A-2}$$

$$\left(\frac{\partial^2 i}{\partial T^2}\right)_P = \left(\frac{\partial c_p}{\partial T}\right)_P = 0 \tag{A-3}$$

$$\left(\frac{\partial^2 i}{\partial P^2}\right)_T = 0 \tag{A-4}$$

$$\frac{\partial^2 u}{\partial v \partial T} = \left(\frac{\partial c_v}{\partial v}\right)_T = \left(\frac{\partial^2 P}{\partial T^2}\right)_v = 0 \tag{A-5}$$

$$\left(\frac{\partial^2 u}{\partial T^2}\right)_v = \left(\frac{\partial c_v}{\partial T}\right)_v = 0 \tag{A-6}$$

Several authors (A-13 - A-17) assume that one single line represents the locus of points of all these maxima. This is indeed the case for subcritical pressure where the saturation line is the locus for all discontinuities. However, the experiments of Kaganer (A-18) and of Sirota and co-workers (A-19) show that this is not the case but that for a given supercritical pressure different thermodynamic properties reach a maximum value at different temperatures. Thus, for each of the thermodynamic properties, i.e., specific heat c_p, the coefficient of thermal expansion, etc., there is a different line which represents the locus of the maxima. This raises the question which of these lines can be regarded to be the extension of the saturation line in the supercritical region, i.e., which of these lines can be considered as the boundary between the liquid-like and the gas-like phase.

Plank (A-20) and Semenchenko (A-21) consider the line along which

$$\left(\frac{\partial^2 p}{\partial v^2}\right)_T = 0 \tag{A-7}$$

to be the extension of the saturation line in the supercritical region. Eucken (A-13), however, takes the curve represented by Eq. A-2 for this extension; whereas numerous authors (A-8, A-9, A-22 - A-25) take Eq. A-3.

Of particular interest to the analysis of this paper are the results reported in (A-14, A-17, A-19 and A-16) which will be therefore discussed in more detail.

Sirota and co-workers (A-19) discuss the transition phenomena at subcritical and supercritical pressures in terms of the Frenkel's theory of heterogeneous fluctuations (A-14, A-26). According to this theory in any gas at subcritical temperature heterogeneous fluctuations result in the formation of molecular complexes which can be regarded as finely dispersed nuclei of a phase within a homogeneous phase. In approaching the saturation line the fluctuations increase and "micro-heterogeneities" appear in the macroscopic, homogeneous phase. This marks the beginning of the "pre-transition region" which is characterized by the fact that various thermodynamic properties exhibit variations which become more pronounced as the saturation line is approached. This accounts for the anomalous effects of the properties in the vicinity of the saturation line. At the saturation line the properties change in a discontinuous fashion which is a characteristic of phase transitions of the first order. As the pressure is increased the effect of heterogeneous fluctuations increases whereas the effect of phase change, i.e., of the discontinuous change of properties becomes less important and disappears at and above the critical point. Since the change of phase at subcritical pressure is characterized by an obsorption of energy and an expansion of volume the transition at supercritical pressure should be characterized by the maximum values of c_p and of the thermal expansion, i.e., of $(\partial v/\partial T)_p$. See Figures A-1 and A-2 which show these properties for oxygen at supercritical pressures. However, the authors of (A-19) show from experiments that at a given pressure the two maxima do not occur at the same temperature. The values of the maxima for C_p are correlated by

$$\frac{c_{p_{max}} - c_{p_g}}{R} = \frac{9.05}{\frac{P}{P_{crit}} - 1} + 1.30 \qquad \text{(A-8)}$$

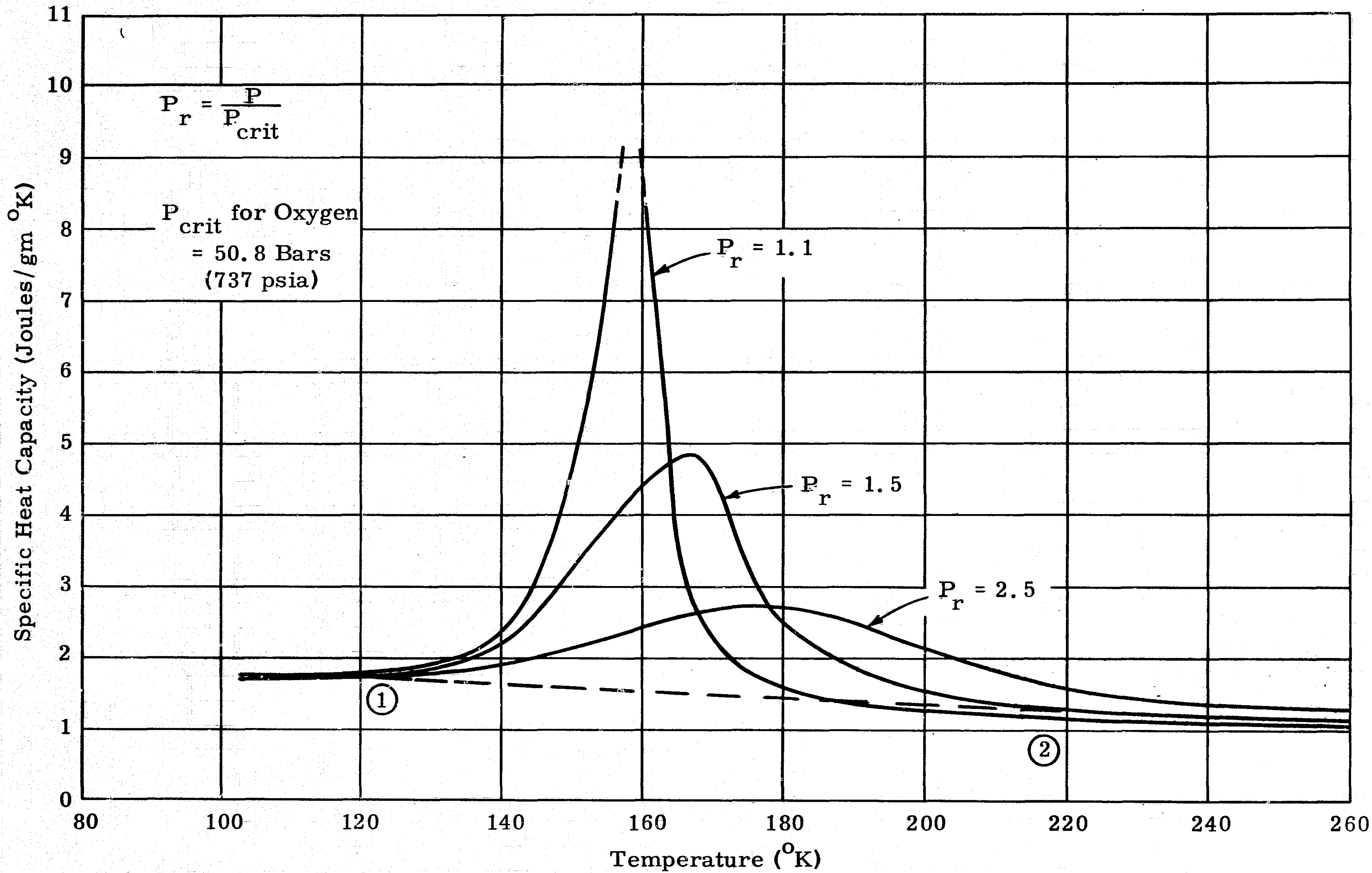

Figure A1 Specific Heat Versus Temperature for Oxygen at Supercritical Pressures.

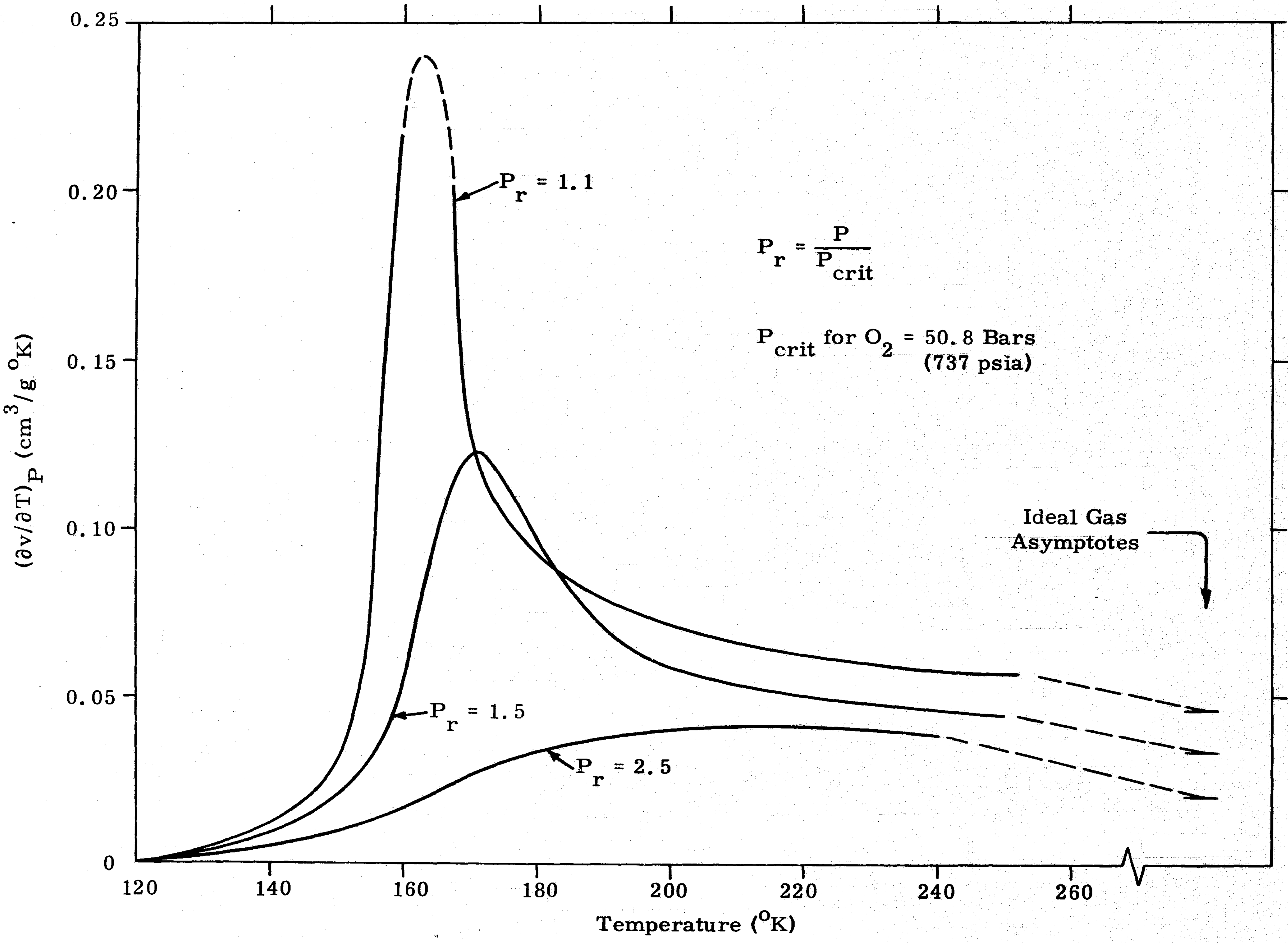

Figure A2 $(\partial v/\partial T)_P$ Versus Temperature for Oxygen at Supercritical Pressures.

which is valid for non-polar liquids when P/P_{crit} 1.5. In the above equation, R is the gas constant whereas c_{p_g} is the specific heat for an ideal gas. This equation shows that the value of the maximum c_p decreases as the pressure is increased. The temperatures where these maxima occur were correlated by

$$\frac{1}{T_{tc}} = \frac{1}{T_{crit}} - 0.487 \log \frac{P}{P_{crit}} \tag{A-9}$$

This temperature, denoted here by T_{pc}, is often referred to in the literature as either the <u>pseudo-critical temperature</u> or the <u>transposed critical temperature</u>.

Both Sirota (A-19) and Kaganer (A-18) show that the locus of the maximum values of c_p along isobars, i.e., Eq. A-3, is the extension of the saturation line in the supercritical region.

Urbakh (A-17) also considers the effect of heterogeneous fluctuations at subcritical and supercritical pressures. He shows that as the temperature is increased and the surface tension decreases the heterogeneous fluctuations increase and reach a maximum at the critical point. The location of the critical point depends on the surface tension; moreover, it can be changed by introducing surface active agents. The critical point divides two regions which can be distinguished by the nature of the phase transition. At subcritical pressure the transition is characterized by the discontinuities of the properties and by the presence of a macroscopic second phase within the originally homogeneous phase. At supercritical pressures the second phase is finely dispersed in the form of clusters. Furthermore, in this region the properties do not change discontinuously but vary in a <u>continuous</u> way. At subcritical pressures the effect of heterogeneous fluctuations becomes evident in the "pretransition region" as a variation of properties in the vicinity of the saturation line. This is shown in Figure A-3 which is the volume-temperature plane for oxygen. At a subcritical pressure, say at $P_r = 0.9$, the line 1' - 2' is the phase transition of the first order occurring at a constant temperature. The effect and magnitude of the fluctuation in specific volume in the two pre-transition regions is shown as the lines 1 - 1' and 2 - 2'. The fluctuation 1 - 1' is caused by the formation of vapor nuclei in the pre-transition region of the liquid. Similarly, 2 - 2' are the fluctuations caused by the formation of liquid nuclei in the pre-transition region of the gas. It can be seen from this Figure that at low pressures in the subcritical region the effect of fluctuation is negligible when compared to the phase transition of the first order. For example, at $P_r = 0.5$, they are almost absent. Increasing the pressure increases the effect of heterogeneous fluctuations which reach a maximum at the critical point. At this point and above it, the phase transition of the first order vanishes so that only the effect of heterogeneous fluctuations remains. Urbakh notes further than with the phase transition and the fluctuations are associated energy requirements which can be determined from the T - s or v - s diagrams shown on Figures A-4 and A-5. At low pressure the only energy required is heat of vaporization for the phase transition of the first order, thus

$$h_{fg} = T\,(s_2' - s_1') \qquad \text{(A-10)}$$

However, as the pressure is increased the energy associated with the fluctuation becomes important. At supercritical pressure it is the only which remains, and it can be determined either from Figure A-4 or A-5, thus

$$\Delta i = T\,(s_2 - s_1) \qquad \text{(A-11)}$$

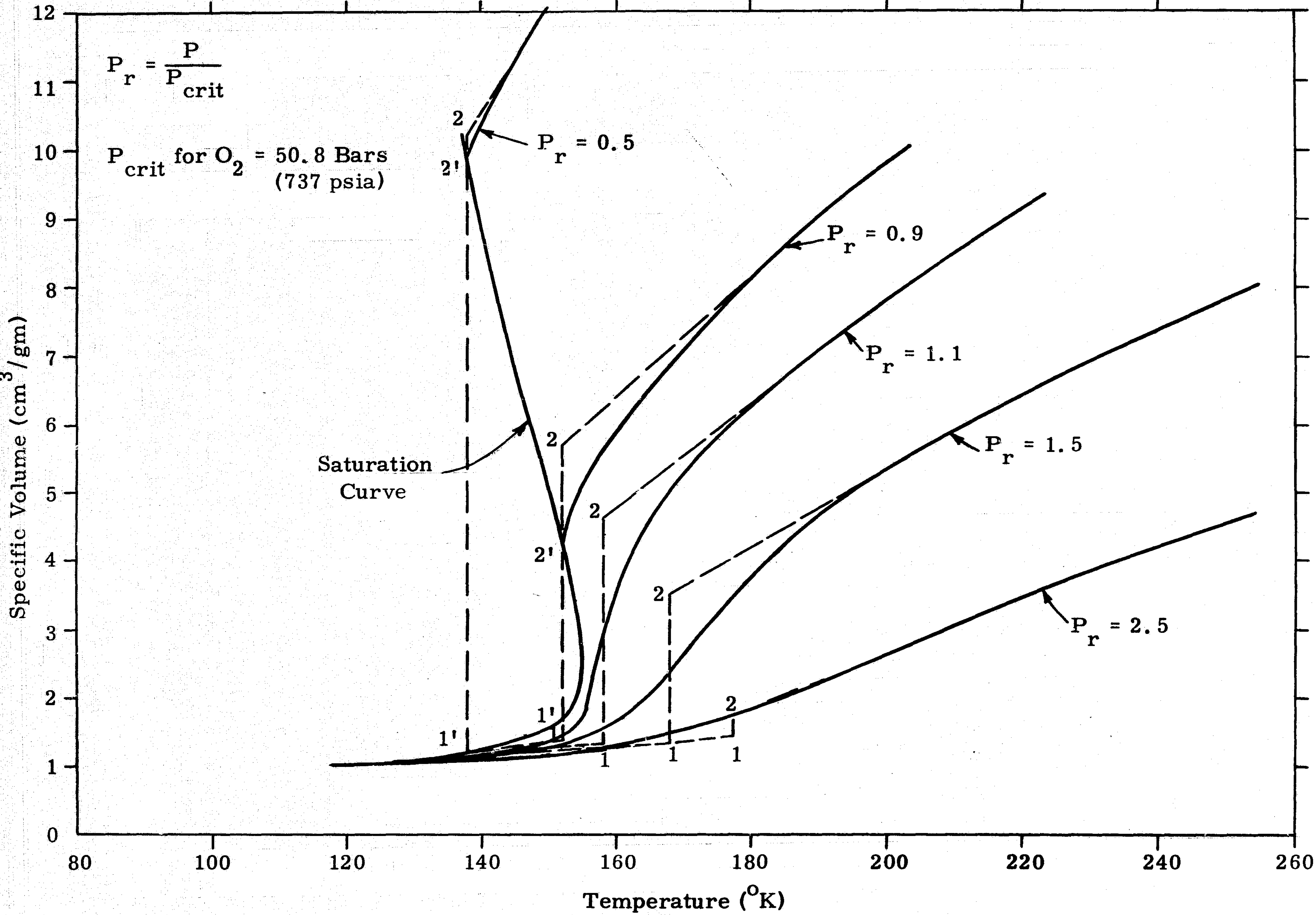

Figure A3 Specific Volume Versus Temperature for Oxygen.

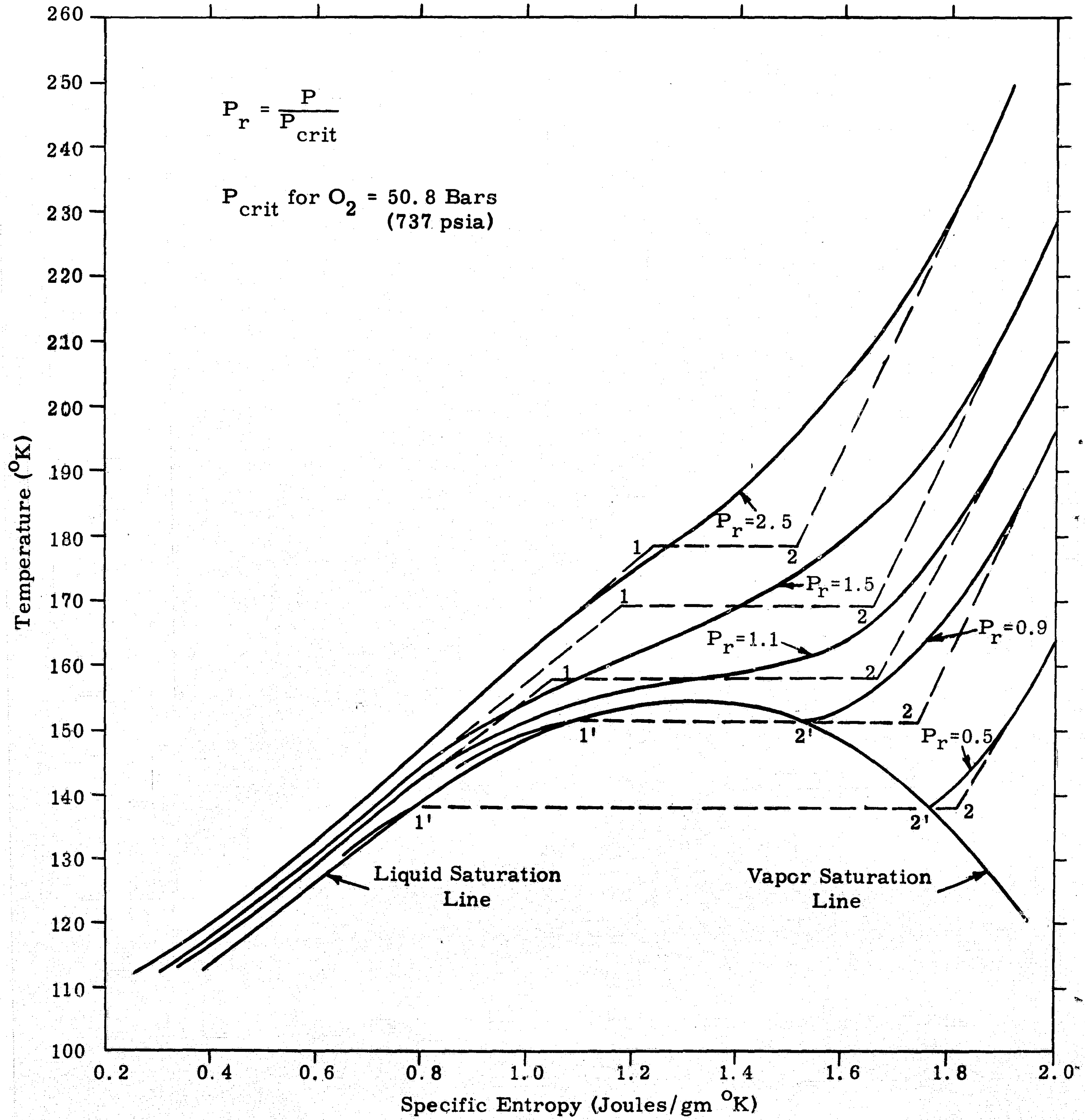

Figure A4 Temperature-Entropy Diagram for Oxygen.

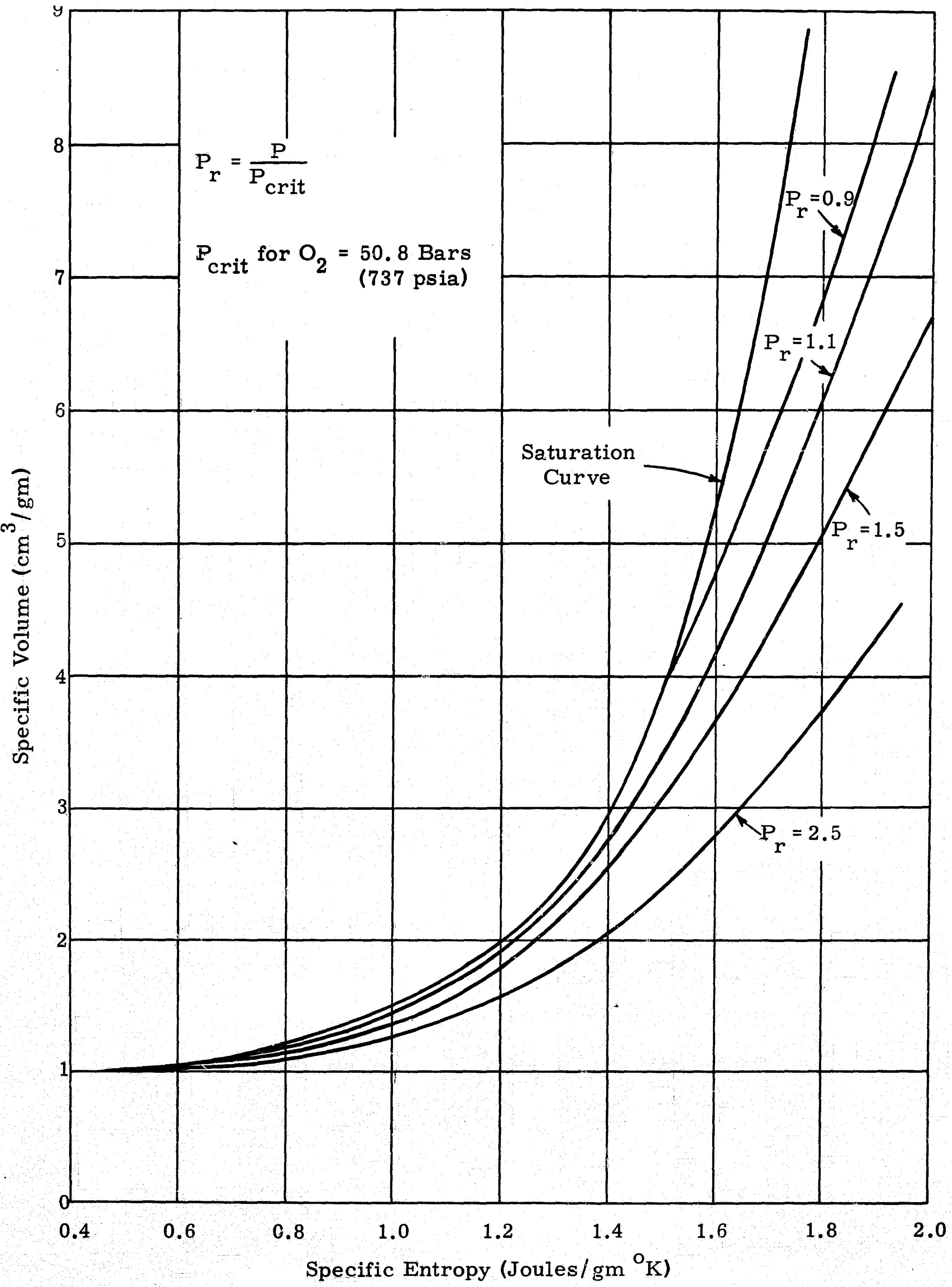

Figure A5 Specific Volume Versus Entropy for Oxygen.

Frenkel (A-14, A-26) considers two variations: a transition of the first order at subcritical pressure and a transition of the second order at supercritical pressure. The first, characterized by discontinuities of properties, is described by Clausius-Clapeyron's equation:

$$\frac{dP}{dT} = \frac{h_{fg}}{T_o\ (v_g - v_f)} \tag{A-12}$$

and takes place at a constant temperature T_o. The phase transition of the second order takes place over a temperature interval $\Delta T = T_2 - T_1$, in which the properties change continuously. In this temperature interval both c_p and $(\partial v/\partial T)_p$ reach a maximum. Figures A1 and A2 show these variations for oxygen at three supercritical pressures. As a generalization of the transition of the first order Frenkel formulates the equivalent energy of transition for the second order transition, thus

$$\Delta i = T_{tc}(s_2 - s_1) = \int_{T_1}^{T_2} \Delta c_p \, dT \tag{A-13}$$

where T_{tc} is the temperature corresponding to the peak of c_p and Δc_p is the value of c_p above the "normal" value, i.e., above the dashed line on Figure 1. Similarly, the change of volume is given by

$$v_2 - v_1 = \int_{T_1}^{T_2} \left(\frac{\partial v}{\partial T}\right)_p dT \tag{A-14}$$

Eq. A-13 and Eq. A-14 represent the additional increase of volume and the additional heat absorbed in going from the liquid-like state to the gas-like state at constant pressure. In place of Clausius Clapeyron's equation Frenkel uses the equation derived by Ehrenfest (A-27) to describe transitions of the second order at the "lambda point" of helium and at the "Curie point" of feromagnetic metals, thus

$$\frac{dP}{dT} = \frac{\Delta c_m}{T_{tc} \Delta \left(\frac{\partial v}{\partial T} \right)_P} \tag{A-15}$$

where Δc_m and $\Delta (\partial v/\partial T)_p$ are the maximum values of c_p and of $(v/ T)_P$ above the dashed lines in Figures A1 and A2. Various criticisms which have been made with respect to Ehrenfest equation are discussed in (A-28). Also, various authors (A-18, A-19) criticize the use of Eq. A-15 for the supercritical region because the temperatures where c_p and $(\partial v/\partial T)_P$ reach their respective maximum values are not the same. Consequently, the value of T_{tc} in Eq. A-15 is somewhat arbitrary.

Semenchenko (A-4, A-16, A-29) considers the medium in the supercritical region to consist of two phases which are separated by a region in which the properties change rapidly but continuously. It was already noted that he takes the locus of points given by Eq. A-7 to represent the extension of the saturation line in the supercritical region. He notes that at subcritical pressures the phase transition is accomplished by absorbing an amount of energy given by Eq. A10 and by doing an amount of work given by:

$$W = P (v_2 - v_1) \tag{A-16}$$

However, since in the supercritical region there is no discontinuous change of volume and of entropy, Semenchenko notes that Eq. A-10 and A-15 must be modified and replaced by:

$$\Delta i = \int_{T_1}^{T_2} c_p \, dT \tag{A-17}$$

and

$$W = \int_{v_1}^{v_2} \left(\frac{\partial p}{\partial v}\right)_T dv \tag{A-18}$$

For additional discussion of critical phenomena the reader is referred to the extensive reviews by Rice (A-1) and by Hammell (A-2).

From the preceding review of the present understanding of thermodynamic phenomena in the supercritical region we can make the following conclusions:

1) There is no general agreement as to the structure of the medium and of the mechanism of phase transition in the critical and supercritical region.

2) There is a general agreement that large variations of density and of specific heat are present.

3) Most of the authors consider the supercritical region to consist of two phases -- a liquid-like and a gas-like phase.

4) There is no general consensus as to the location of the boundary or of the transition region between these two phases, although a large number of investigators consider this demarkation to take place along the line which is the locus of points where the specific heat at constant pressure reaches a maximum.

5) There is no general consensus as to the nature of phase transition at supercritical pressures and of the energy required to bring it about. Three different methods for evaluating this energy of transition have been proposed: 1) the graphical method of Urbakh (A-17) resulting in Eq. A-11; 2) the second order transition proposed by Frenkel (A-14, A-26) given by Eq. A-13, and Eq. A-15; and 3) the pseudo transition region proposed by Semenchenko (A-4, A-16, A-29) given by Eq. A-15 and Eq. A-17. By examining the proposed methods and equations, i.e., Eq. A-11, Eq. A-15 and A-17, it can be seen that these different methods will yield different values for the transition energy.

It is evident from the preceding results that the success of any analysis concerned with the mechanism of flow oscillations and of heat transfer at supercritical pressures will depend to a great extent upon the ability to describe more accurately the thermodynamic state of a fluid and the transition phenomena that take place at supercritical pressures.

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LIST OF ILLUSTRATIONS - APPENDIX A

Appendix B

The Steady State Pressure Drop

In this Appendix we shall derive an expression for the steady state pressure drop of a fluid whose properties change from a liquid-like at the entrance to a gas-like at the exit of the heat exchanger. The derivation and the resulting flow excursion criterion applicable to fluids at critical and supercritical pressures were first derived by the writer in the Second Quarterly Progress Report. They are reproduced here for reasons of completeness.

The pressure drop across a heated length L is the sum of the acceleration, pressure drop, the frictional pressure drop and the pressure drops across the inlet and exit flow restrictions. Since the pressure drop depends on the fluid, it becomes necessary to examine first property changes along the heated duct.

B.1 The System - Three Region Approximation

The system analyzed in this Appendix is shown in the Figure B-1.

A circular duct is uniformly heated at a rate of $\dot{Q}$, over a total heated length L. Two flow restrictions are located at the entrance and at the exit of the heated section. A fluid at an initial temperature T_1, i.e., with the enthalpy i_1, flows at a constant mass flow rate w. In passing through the heated duct the specific volume and the enthalpy of the fluid increase (See Fig. B-1). The fluid undergoes, therefore, a transformation from a liquid-like to a gas-like fluid.

Figure B-2 shows the v-i relation for oxygen at a reduced pressure of $P_r = 1.1$. It can be seen from this figure that the increase of specific

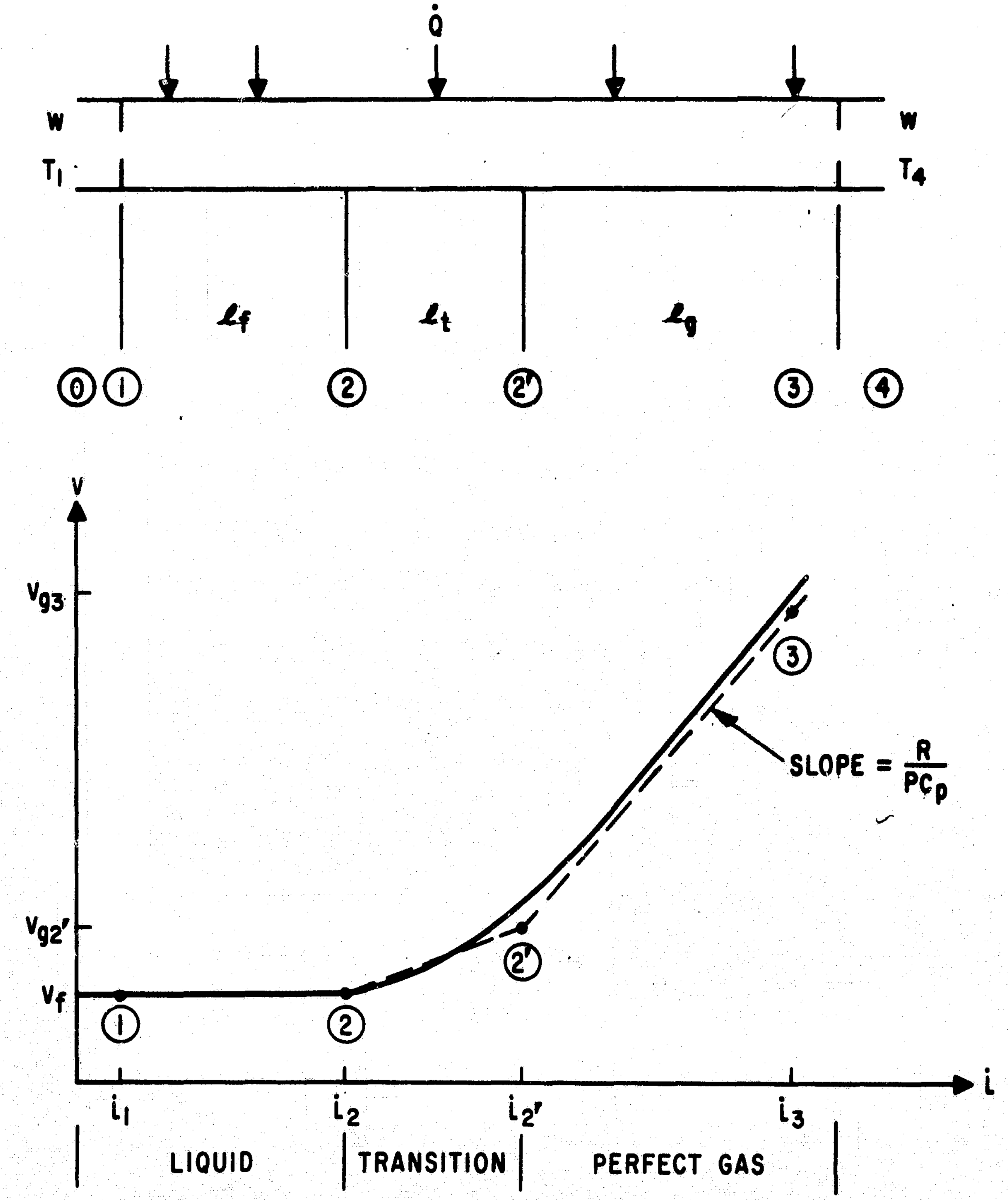


FIGURE B1 THE SYSTEM FOR A THREE-REGION APPROXIMATION volume from a liquid-like state to a gas-like state occurs gradually over an enthalpy interval.

In order to simplify the problem, we shall assume that the entire transformation can be approximated by considering three regions. In the first region of length l_f, between stations (1) and (2) in Figure B-1, the heavy clusters resemble a liquid. In this region the specific volume of the fluid is constant having a value of v_f. We shall assume that the complete transformation, from heavy to light clusters, takes place within the transition length l_t, i.e., between stations (2) and (2') . In this transition region the specific volume of the fluid changes from a value of v_f to a value of v_{g2} . The enthalpy change associated with this expansion is given by $\Delta i_{2'2} = i_{2'} - i_2$. In the third region of length l_g, the light clusters resemble a gas. The specific volume of the fluid in this region can be approximated by that of gas and, in particular, by that of a perfect gas.

It is apparent from the discussion in Appendix A that the initial and the final conditions of the transition region, i.e., the conditions at stage (2) and (2') respectively, will depend upon the model selected for describing the pseudo-phase transition in the supercritical region. This follows from the fact that the temperature or the enthalpy that marks the start of the pseudo-phase transition will determine the location of station (2), whereas the location of station (2') will depend on the energy required to complete the transition from heavy to the light clusters. In this report we shall denote this energy requirement by $\Delta i_{2'2}$ which can be determined by the best three region approximation indicated in Figure B-1.

As discussed in the preceeding sections, we are considering in this report only the effects of density variation on the flow stability.

Consequently, we shall assume that both the friction factor and the heat transfer are constant. The first assumption is quite reasonable if the flow remains turbulent throughout the duct. The limitation of the second assumption may become significant if variations of transport properties in the transition region have an important effect on the stability. We note, however, that both assumptions can be removed permitting an extension of the analysis to consider the effect of variations, other than density, on the initiation of flow oscillations.

B.2 <u>The Frictional Pressure Drop</u>

The frictional pressure drop in the system is given by the sum of the frictional pressure drops across the segments l_f, l_t and l_g and the pressure drops across the inlet and exit flow restrictions, thus

$$\Delta P(W) = \Delta P_{12}(W) + \Delta P_{22'}(W) + \Delta P_{2'3}(W) \qquad \text{B-1}$$

For a constant friction factor f, the pressure drop across a segment of length l is given by

$$\Delta P_l(W) = \frac{f}{2D}\left(\frac{W}{A}\right)^2 \bar{v}\, l \qquad \text{B-2}$$

where the lengthwise average specific volume is given by

$$\bar{v} = \frac{1}{l}\int_0^l v\, dz \qquad \text{B-3}$$

Consequently, in order to evaluate the frictional pressure drops for the three segments it is necessary to evaluate the specific volume for each segment. This can be done by relating first the specific volume to enthalpy and then to express the enthalpy in terms of the heated length. This latter relation can be obtained from energy considerations.

Denoting by $\dot{Q}$, the total rate of energy addition to the system and by q the constant heat flux density, we have for a duct

$$\frac{dQ}{dz} = q\xi \qquad \text{B-4}$$

where ξ, is the heated perimeter. It follows from Eq. B-4 that

$$\frac{Q}{L} = q\xi \qquad \text{B-5}$$

where the total length is given by

$$L = l_f + l_t + l_g \qquad \text{B-6}$$

Furthermore, for a system with constant mass flow rate the change of enthalpy is given by

$$W\,di = d\dot{Q} \qquad \text{B-7}$$

where we have neglected the kinetic energy of the fluid. It follows then from Eq. B-7 and B-4 that

$$W\,di = q\xi\,dz \qquad \text{B-8}$$

and in view of Eq. B-5 we obtain

$$\frac{L}{\dot{Q}} W di = dz \qquad \text{B-9}$$

Substituting Eq. B-4, B-3 in Eq. B-2, we obtain the pressure drop across a heated segment where the enthalpy of the fluid changes from i to i + Δi, thus

$$\Delta P = \frac{fL}{2D} \frac{W}{\dot{Q}} \left(\frac{W}{A}\right)^2 \int_{i}^{i+\Delta i} v(i)\, di \qquad \text{B-10}$$

For a three region approximation the relation between v(i) and i is shown in Figure B-1. We shall consider now each region separately.

a) The Liquid-Like Region

In this region the specific volume of the fluid is constant and equal to v_f (See Figure B-1). In the segment of length l_f, the enthalpy of the fluid increases from i_1 to i_2. The frictional pressure drop across l_f becomes then

$$\Delta P_{12} = \frac{fL}{2D}\left(\frac{W}{A}\right)^2 \frac{W}{\dot{Q}} \Delta i_{21} v_f \qquad \text{B-11}$$

b) The Transition Region

In the transition region we shall approximate the relation between the specific volume v, and the enthalpy i, by a linear equation. The average specific volume in this region can be written then as:

$$\bar{v}_{2'2} = v_f + \frac{v_{g2'} - v_f}{2} = v_f + \frac{\Delta v_{fg}}{2} \qquad \text{B-12}$$

Denoting by $\Delta i_{2'2} = i_{2'} - i_2$ the change in enthalpy, the frictional pressure drop in the transitional region then becomes

$$\Delta P_{22'} = \frac{fL}{2D}\left(\frac{W}{A}\right)^2 \frac{W}{\dot{Q}} \Delta i_{2'2}\left(v_f + \frac{\Delta v_{fg}}{2}\right) \tag{B-13}$$

c) The Gas-Like Region

In view of the assumption that in this region the fluid has the properties of a perfect gas we have, for a constant pressure process, the following expression for the specific volume

$$v_g = v_{g2'} + \frac{R}{Pc_p}(i - i_{2'}) \tag{B-14}$$

Inserting this expression in Eq. B-10 we obtain

$$\Delta P_{2'3} = \frac{fL}{2D}\left(\frac{W}{A}\right)^2 \frac{W}{\dot{Q}} \Delta i_{32'}\left(v_{g2'} + \frac{1}{2}\frac{R}{Pc_p}\Delta i_{32'}\right) \tag{B-15}$$

The change of enthalpy Δi_{32} , can be expressed also in terms of the total heat input thus from an energy balance

$$\Delta i_{32'} = \frac{\dot{Q}}{W} - \Delta i_{2'2} - \Delta i_{21} \tag{B-16}$$

Inserting this expression in Eq. B-15, we obtain for the frictional pressure drop in the gas-like region the following expression

$$\Delta P_{2'3} = \frac{fL}{2D}\left(\frac{W}{A}\right)^2 \frac{W}{\dot{Q}}\left[\frac{\dot{Q}}{W} - \Delta i_{2'2} - \Delta i_{21}\right]\left[v_{g2'} + \frac{1}{2}\frac{R}{Pc_p}\left(\frac{\dot{Q}}{W} - \Delta i_{22'} - \Delta i_{21}\right)\right] \tag{B-17}$$

B.3 The Inlet and Outlet Pressure Drops

Denoting by k_i a numerical coefficient that takes into account the geometry of the restriction and of other losses like vena contracta etc., we have the inlet pressure drop

$$\Delta P_{01} = k_i v_f W^2 \tag{B-18}$$

Similarly, we define by k_e a numerical coefficient that accounts for the geometry and the losses at the exit. The exit pressure can be then expressed as:

$$\Delta P_{34} - k_e v_{g3} W^2 \tag{B-19}$$

which, in view of Eq. B-14 and B-16, can be also written as:

$$\Delta P_{34} = k_e W^2 \left[v_{g2'} + \frac{1}{2} \frac{R}{P c_p} \left(\frac{\dot{Q}}{W} - \Delta i_{2'2} - \Delta i_{21} \right) \right] \tag{B-20}$$

B.4 The Acceleration Pressure Drop

The acceleration pressure drop is given by

$$dp = \rho u \, du \tag{B-21}$$

as

$$\frac{W}{A} = \rho u = \frac{u}{v} \tag{B-22}$$

then

$$dp = \frac{W}{A}\, dv \qquad \text{B-23}$$

The total acceleration pressure drop required to accelerate a fluid of specific volume v_f at the inlet up to the exit where it attains a specific volume v_{g3} is given therefore by

$$\Delta P_a = \left(\frac{W}{A}\right)^2 (v_{g3} - v_f) \qquad \text{B-24}$$

Inserting Eq. B-14 in Eq. 24 we obtain

$$\Delta P_a = \left(\frac{W}{A}\right)^2 \left[v_{g2'} + \frac{R}{P c_p} \Delta i_{32'} - v_f \right] \qquad \text{B-25}$$

or in view of Eq. B-16 we have

$$\Delta P_a = \left(\frac{W}{A}\right)^2 \left[v_{g2'} + \frac{R}{P c_p} \left(\frac{\dot{Q}}{W} - \Delta i_{2'2} - \Delta i_{21} \right) - v_f \right] \qquad \text{B-26}$$

B.5 The Total Pressure Drop

The total pressure drop is obtained by summing Eq. B-26, B-20, B-18, B-17 and B-11, thus after some rearrangement

$$\Delta P_a + \Delta P_{04} = a \frac{W^3}{\dot{Q}} - b W^2 + c \dot{Q} W \qquad \text{B-27}$$

where the coefficients a, b and c are given by

$$a = \frac{fL}{2DA^2}\left\{ v_f \Delta i_{21} + \left(v_f + \frac{\Delta v_{fg}}{2}\right)\Delta i_{2'2} + \right.$$

$$\left. + \left[\frac{1}{2}\frac{R}{Pc_p}\left(\Delta i_{21} + \Delta i_{2'2}\right) - v_{g2'}\right]\left[\Delta i_{21} + \Delta i_{2'2}\right] \right\} \quad \text{B-28}$$

$$b = \frac{fL}{2DA^2}\left\{ \frac{2D}{fL}\left[\frac{R}{Pc_p}\left(\Delta i_{21} + \Delta i_{2'2}\right) - v_{g2'} + v_f\right)\right] + $$

$$+ \frac{R}{Pc_p}\left(\Delta i_{21} + \Delta i_{2'2}\right) - v_{g2'} - k_i \frac{2DA^2}{fL} v_f + \quad \text{B-29}$$

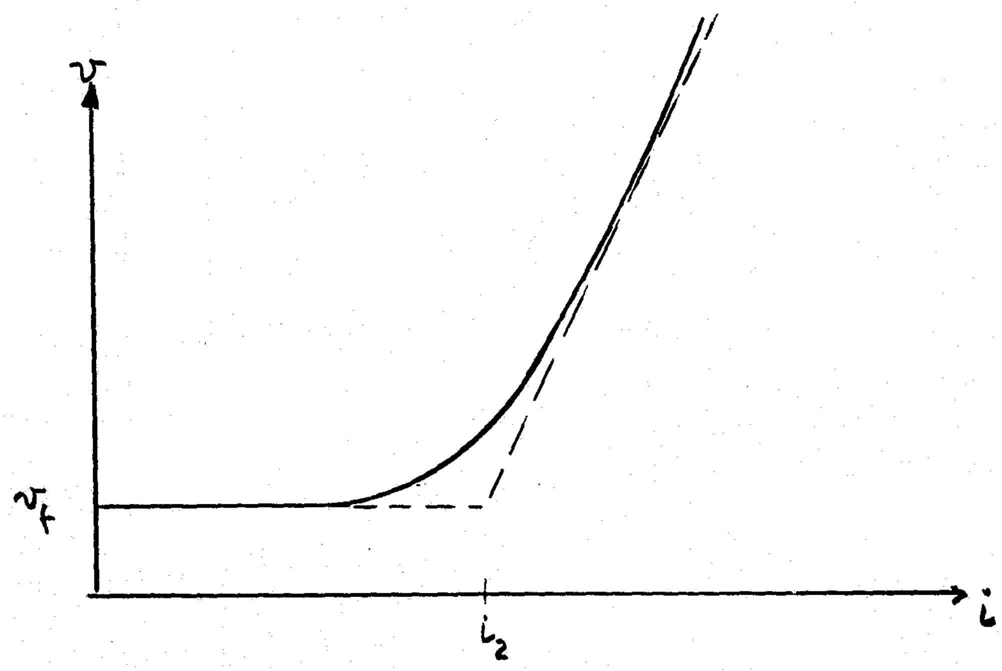
$$\left. + k_e \frac{2DA^2}{fL}\left[\frac{1}{2}\frac{R}{Pc_p}\left(\Delta i_{21} + \Delta i_{2'2}\right) - v_{g2}\right] \right\}$$

$$c = \frac{fL}{2DA^2}\left[\frac{2D}{fL}\, 2 + k_e \frac{2DA^2}{fL} + 1\right]\frac{1}{2}\frac{R}{Pc_p} \quad \text{B-30}$$

The form of Eq. B-27 is relevant to the present problem because it shows that, for some operating conditions, the pressure drop may decrease with increasing mass flow. This consequence of the negative term on the right-hand side of Eq. B-27.

B.6 The Two Region Approximation

Following the derivation of the pressure drop given in the preceding section it was observed by Dr. R. Fleming, from the Research and Development Center at G.E., that instead of considering a three region approximation as shown in Fig. B-1, the problem can be further simplified by considering a two-region approximation indicated in the sketch below



Two-Region Approximation

In the two-region approximation the transition region shown in Fig. B-1 and B-2 is neglected, i.e., $l_t = 0$, $i_2 = i_2'$, $v_{g2'} = v_f$. It is assumed, therefore, that the change from a liquid-like to a gas-like fluid occurs instantaneously in a plane perpendicular to the flow when the enthalpy reaches a value of i_2 indicated on the sketch above.

We can further amplify the preceding observation. It can be seen from Fig. B-2 that the enthalpy which corresponds to the <u>transition point</u> can be approximated by the enthalpy at the <u>transposed critical temperature</u> T_{tc}, i.e., by the enthalpy that corresponds to the <u>maximum</u> value of the specific heat at constant pressure c_p. Consequently, with a two-region approximation one can consider that the liquid-like state persists until the temperature of the bulk fluid reaches a value that is equal to the transposed critical (or pseudo-critical) temperature. Above that temperature the fluid behaves as a gas. Therefore, the transposed critical temperature T_{tc}, can be regarded as the boundary between the liquid-like and the gas-like states. It was discussed already in Appendix A, that both Sirota (A-19) and Kaganer (A-18) have shown that this temperature is the extension of the saturation line in the supercritical region. Figure A-1 in Appendix A shows that the transposed critical temperature increases with increasing values of reduced pressure. It can be concluded therefore that the value of enthalpy corresponding to this temperature and to the transition point shown in Fig. B-2 will also increase with increasing reduced pressures.

For a two-region approximation the form of Eq. B-27 remains unchanged, however, the coefficients a, b and c given by Eq. B-28 and B-29 and B-30 reduce to:

$$a = \frac{fL}{2DA^2}\left\{\frac{1}{2}\frac{R}{Pc_p}\Delta i_{21}^2\right\} \qquad \text{B-31}$$

$$b = \frac{fL}{2DA^2}\left\{\frac{2D}{fL}\frac{R}{Pc_p}\Delta i_{21} + \frac{R}{Pc_p}\Delta i_{21} - v_f\left(1 + K_i\frac{2DA^2}{fL}\right) + \right. \qquad \text{B-32}$$

$$\left. + k_e\frac{2DA^2}{fL}\left(1 + \frac{R}{Pc_p}\Delta i_{21} - v_f\right)\right\}$$

$$c = \frac{fL}{2DA^2}\left\{\frac{4D}{fL} + k_e\frac{2DA^2}{fL} + 1\right\}\frac{1}{2}\frac{R}{Pc_p} \qquad \text{B-33}$$

which we obtain by setting $\Delta i_{2'2} = 0$, $\Delta v_{fg} = 0$, $V_{g2'} = v_f$ in Eq. B-28, B-29 and B-30.

As noted by Dr. Fleming the use of the two-region approximation simplifies considerably the form of the coefficients a, b and c. The three region approximation retains however a closer similarity with phenomena that take place at subcritical pressure. The transition region shown in Fig. B-1 can be regarded as corresponding to the boiling region at subcritical pressures. The liquid and the gas region in Fig. B-1 would then correspond to the pre-heating and to the superheating region in a once-through boiling system where the liquid at the entrance is subcooled and the steam at the exit is superheated. We have noted already in Appendix A that the enthalpy change $\Delta i_{2'2}$ may be considered as being equivalent to the heat of vaporization h_{fg}.

The selection of either the two or three region approximation should be determined by the desired simplicity and accuracy. The important result is however the fact that, because of the negative term on the right-hand side of Eq. 27, there exist a possibility of a decrease in pressure drop with increasing flow in the supercritical thermodynamic region. It was shown in the body of the report that such a pressure drop vs flow relation can lead either to excursive flow or to oscillatory flow.

Appendix C

The upper and lower bounds of the integrals

The integrals given by Eq. IV-94 , IV-101 and IV-111 can be all expressed in the form of

$$I = K \frac{1}{\frac{\bar{u}_3}{\bar{u}_1} - 1} \int_1^{\bar{u}_3/\bar{u}_1} \left[\frac{\bar{u}_1}{u_q(z)} \right]^m d\left[\frac{u_q(z)}{\bar{u}_1} \right] \qquad \text{C-1}$$

which integrates in

$$I = K \frac{1}{-m+1} \frac{1}{\frac{\bar{u}_3}{\bar{u}_1} - 1} \left\{ \left(\frac{\bar{u}_1}{\bar{u}_3} \right)^{m-1} - 1 \right\} \qquad \text{C-2}$$

where K is a coefficient and m an exponent. For example, the integral given by Eq. IV-111 is

$$I_s = \frac{f \rho_f}{2D} \int_{\lambda(t)}^{l} \frac{\Omega}{s-\Omega} \left\{ \left[\frac{\bar{u}_1}{\bar{u}_q(z)} \right]^{s/\Omega} e^{-s\tau_b} \frac{\delta u_1}{\bar{u}_q(z)} \right\} u_q^2(z)\, dz \qquad \text{C-3}$$

It can be expressed by

$$I_s = \frac{f(l-\bar{\lambda})}{2D} \rho_f \frac{\Omega}{s-\Omega} e^{-s\tau_b} \delta u_1 \frac{\Omega}{s-\Omega} u_1 \int_{\lambda(t)}^{l} \left[\frac{\bar{u}_1}{\bar{u}_q(z)} \right]^{\frac{s}{\Omega}-1} dz \qquad \text{C-3}$$

However, in view of Eq. IV-34 and IV-28 we have

$$\Omega(l-\bar{\Lambda}) = \bar{u}_3 - \bar{u}_1 \qquad \text{C-4}$$

and

$$\bar{u}_g(z) = \bar{u}_1 + \Omega(z-\bar{\Lambda}) \qquad \text{C-5}$$

$$d\,\bar{u}_g(z) = \Omega\, dz \qquad \text{C-6}$$

whence we can express Eq. C-3 as

$$I_s = \frac{f(1-\bar{\Lambda})}{2D} \rho_f \frac{\Omega}{s-\Omega} e^{-s\tau_b} \delta u_1 \frac{u_1}{\frac{\bar{u}_3}{\bar{u}_1}-1} \int_1^{\bar{u}_3/\bar{u}_1} \left[\frac{\bar{u}_1}{\bar{u}_g(z)}\right]^{\frac{s}{\Omega}-1} d\left[\frac{\bar{u}_g(z)}{\bar{u}_1}\right] \qquad \text{C-7}$$

By comparing Eq. C-7 with Eq. C-1 it can be seen that they are of the same form.

In view of Eq. C-2, the integration of Eq. C-7 yields:

$$I_s = \frac{f(1-\bar{\Lambda})}{2D} \rho_f \frac{\Omega}{s-\Omega} e^{-s\tau_b} \bar{u}_1 \delta u_1 \frac{1}{-\frac{s}{\Omega}+2} \frac{1}{\frac{\bar{u}_3}{\bar{u}_1}-1} \left\{\left[\frac{\bar{u}_1}{\bar{u}_3}\right]^{\frac{s}{\Omega}-1} - 1\right\} \qquad \text{C-8}$$

which, after some rearrangement, can be expressed in the form of Eq. IV-111, thus

$$I_s = \frac{f}{2D} \rho_f \bar{u}_1^2 \frac{\Omega}{s-\Omega} \frac{\Omega}{s-2\Omega} \frac{e^{-s\tau_s}}{\Omega} \left\{ 1 - \left(\frac{\bar{u}_1}{\bar{u}_3} \right)^{\frac{s}{\Omega}-2} \right\} \qquad \text{C-9}$$

In order to obtain the upper and lower bound of Eq. IV-94, IV-101, IV-111 we note that Eq. C-1 can be written as

$$I = K \bar{F} \qquad \text{C-10}$$

where $\bar{F}$ is the mean value of F given by

$$\bar{F} = \frac{1}{b-a} \int_a^b F(x)\, dy = \frac{1}{\frac{\bar{u}_3}{\bar{u}_1} - 1} \int_1^{\bar{u}_3/\bar{u}_1} \left[\frac{\bar{u}_1}{u_g(3)} \right]^m d\left[\frac{\bar{u}_g(3)}{\bar{u}_1} \right] \qquad \text{C-11}$$

whence by the mean value thereom

$$\left[\frac{\bar{u}_1}{\bar{u}_3} \right]^m < \bar{F} < 1 \qquad \text{C-12}$$

which together with Eq. C-10 yields the upper and lower bounds given in Eq. IV-95, IV-105 and IV-116, thus

$$K\left(\frac{\bar{u}_1}{\bar{u}_3}\right)^m < I < K \qquad \text{C-13}$$

For example, from Eq. C-7 and Eq. C-12 we obtain

$$\frac{f(1-\bar{\lambda})}{2D} G \frac{\Omega}{s-\Omega} e^{-s\tau_b} \left[\frac{\bar{u}_1}{\bar{u}_3}\right]^{s/\Omega} \frac{\bar{u}_3}{\bar{u}_1} < I_s < \frac{f(1-\bar{\lambda})}{2D} G \frac{\Omega}{s-\Omega} e^{-s\tau_b} \qquad \text{C-14}$$

since

$$\left[\frac{\bar{u}_1}{\bar{u}_3}\right]^{s/\Omega} = e^{-s(\tau_3-\tau_2)} \qquad \text{C-15}$$

it can be seen that Eq. C-14 can be put in the form of Eq. IV-116.